

DERIVATIVES FOR REPRESENTATIONS OF $GL(n, \mathbb{R})$ AND $GL(n, \mathbb{C})$

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ABSTRACT. The notion of derivatives for smooth representations of $GL(n, \mathbb{Q}_p)$ was defined in [BZ77]. In the archimedean case, an analog of the highest derivative was defined for irreducible unitary representations in [Sah89] and called the “adduced” representation. In this paper we define derivatives of all order for smooth admissible Fréchet representations (of moderate growth). The real case is more problematic than the p -adic case; for example arbitrary derivatives need not be admissible. However, the highest derivative continues being admissible, and for irreducible unitarizable representations coincides with the space of smooth vectors of the adduced representation.

We prove exactness of the highest derivative functor, and compute highest derivatives of all monomial representations. We apply those results to finish the computation of adduced representations for all irreducible unitary representations and to prove uniqueness of degenerate Whittaker models for unitary representations, thus completing the results of [Sah89, Sah90, SaSt90, GS].

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1. INTRODUCTION

The notion of derivative was first defined in [BZ77] for smooth representations of $G_n = GL(n)$ over non-archimedean fields and became a crucial tool in the study of this category. The purpose of the present paper is to transfer part of this construction to the archimedean case.

The definition of derivative is based on the “mirabolic” subgroup P_n of G_n consisting of matrices with last row $(0, \dots, 0, 1)$. The unipotent radical of this subgroup is an $(n - 1)$ -dimensional linear space that we denote V_n , and the reductive quotient is G_{n-1} . The group G_{n-1} has 2 orbits on V_n and hence also on V_n^* : the zero and the non-zero orbit. The stabilizer in G_{n-1} of a non-trivial character ψ of V_n is isomorphic to P_{n-1} .

The construction of derivative in [BZ77] is based on two functors: Φ^- and Ψ^- . In this paper we denote those functors just by Φ and Ψ . The functor Ψ goes from the category of smooth representations of the mirabolic group P_n to the category of smooth representations of G_{n-1} (for each n) and Φ goes from the category of smooth representations of P_n to the category of smooth representations of P_{n-1} . The functor Ψ is the functor of (normalized) coinvariants with respect to V_n and the functor Φ is the functor of (normalized) co-equivariants with respect to (V_n, ψ) . The functor of k -th derivative is defined to be the composition $\Psi \circ \Phi^{k-1}$.

Another way to describe those two functors is via the equivalence of categories of the smooth representations of P_n and the category of G_{n-1} -equivariant sheaves on V_n^* . This equivalence is based on the

Fourier transform. Under this equivalence, Ψ becomes the fiber at 0 and Φ becomes the fiber at the point ψ . The functor Φ can also be viewed as the composition of two functors: restriction to the open orbit and an equivalence of categories between equivariant sheaves on the orbit and representations of the stabilizer of a point.

In the archimedean case, the notion of fiber of a sheaf behaves differently than in the non-archimedean case; in particular it is not exact. One way to deal with this problem is to consider instead the notion of a stalk or a jet. On the level of representations this means that one uses generalized coinvariants instead of usual coinvariants. For example, the Casselman-Jacquet functor is defined in this way. Therefore in our definition we replace the functor Ψ by the functor of generalized coinvariants. However, we do not change the definition of the functor Φ since we think of it as restriction to an open set followed by an equivalence of categories, and in particular it should be exact.

This gives the following definition of derivative. Let ψ_n be the standard non-degenerate character of V_n , given by $\psi_n(x_1, \dots, x_{n-1}) := \exp(\sqrt{-1}\pi \operatorname{Re} x_{n-1})$. We will also denote by ψ_n the corresponding character of the Lie algebra \mathfrak{v}_n . For all n and for all representations π of \mathfrak{p}_n , we define

$$\Phi(\pi) := |\det|^{-1/2} \otimes \pi_{(\mathfrak{v}_n, \psi_n)} := |\det|^{-1/2} \otimes \pi / \operatorname{Span}\{\alpha v - \psi_n(\alpha)v : v \in \pi, \alpha \in \mathfrak{v}_n\}$$

and

$$\Psi(\pi) := \varprojlim_l \pi / \operatorname{Span}\{\beta v \mid v \in \pi, \beta \in (\mathfrak{v}_n)^{\otimes l}\}.$$

Now, define $D^k(\pi) := \Psi\Phi^{k-1}(\pi)$.

Consider the case $k = n$; in this case the derivative becomes the (dual) Whittaker functor. It is well known that the behavior of the Whittaker functor depends on the category of representations that we consider. For example in the category of Harish-Chandra modules the Whittaker functor gives high dimensional vector spaces while in the equivalent category of smooth admissible Fréchet representations the Whittaker functor gives vector spaces of dimension ≤ 1 just as in the non-archimedean case. For this reason we view the functor D^k restricted to the category of smooth admissible Fréchet representations as the natural counterpart of the Bernstein-Zelevinsky derivative.

Nevertheless in order to study this functor we will need to consider also the category of Harish-Chandra modules as well as some other related functors.

In the non-archimedean case the highest non-zero derivative plays a special role. It has better properties than the other derivatives. In particular in its definition one can omit the last step of Ψ since V_{n-k} already acts trivially on the obtained representation. The index of the highest derivative is called the depth of the representation. As observed in [GS] the depth can also be described in terms of the wavefront set of the representation.

In the archimedean case the wavefront set of a representation π is determined by its annihilator variety, and we will use the latter to define ‘‘depth’’. We recall that if π is an admissible representation of G_n then its annihilator variety \mathcal{V}_π is a subset of the cone of nilpotent $n \times n$ matrices. We define the depth of π to be the smallest index d such that $X^d = 0$ for all $X \in \mathcal{V}_\pi$.

Example.

- (1) For a finite dimensional representation π of G_n , $\operatorname{depth}(\pi) = 1$, $D^1(\pi) = \Phi(\pi)|_{G_{n-1}} = \pi|_{G_{n-1}}$, and $D^k(\pi) = 0$ for any $k > 1$.
- (2) $D^n = (\Phi)^{n-1}$ is the Whittaker functor. On the category of smooth admissible Fréchet representations it is proven to be exact in [CHM00] and in the category of admissible Harish-Chandra modules in [Kos78]. It is also proven in [Kos78] that $\operatorname{depth}(\pi) = n$ if and only if $D^n(\pi) \neq 0$ (in both categories).

From now on, let F be an archimedean local field and $G_n := GL(n, F)$.

In this paper we will mainly be interested in the depth derivative. The following theorem summarizes the main results of this paper.

Theorem A. *Let $\mathcal{M}_\infty(G_n)$ denote the category of smooth admissible Fréchet representations of moderate growth and let $\mathcal{M}_\infty^d(G_n)$ denote the subcategory of representations of depth $\leq d$. Then*

- (1) D^d defines a functor $\mathcal{M}_\infty^d(G_n) \rightarrow \mathcal{M}_\infty(G_{n-d})$.

- (2) The functor $D^d : \mathcal{M}_\infty^d(G_n) \rightarrow \mathcal{M}_\infty(G_{n-d})$ is exact.
- (3) For any $\pi \in \mathcal{M}_\infty^d(G_n)$, $D^d(\pi) = (\Phi)^{d-1}(\pi)$.
- (4) $D^k|_{\mathcal{M}_\infty^d(G_n)} = 0$ for any $k > d$.
- (5) Let $n = n_1 + \cdots + n_d$ and let χ_i be characters of G_{n_i} . Let $\pi = \chi_1 \times \cdots \times \chi_d \in \mathcal{M}_d(G_n)$ denote the corresponding monomial representation. Then

$$D^d(\pi) \cong ((\chi_1)|_{G_{n_1-1}} \times \cdots \times (\chi_d)|_{G_{n_d-1}})$$

- (6) If τ is an irreducible unitary representation of G_n and τ^∞ has depth d then $D^d(\tau^\infty) \cong (A\tau)^\infty$, where $A\tau$ denotes the adduced representation defined in [Sah89] (see §§4.1.3).

Remark.

- (i) Part (3) of the theorem means that \mathfrak{v}_{n-d+1} acts nilpotently on $(\Phi)^{d-1}(\pi)$. Unlike the p -adic case, V_{n-d+1} need not act trivially on $(\Phi)^{d-1}(\pi)$.
- (ii) In this paper we do not prove that for π of depth d , $D^d(\pi) \neq 0$. This is in fact true but needs an additional argument which will be provided in [GS2]. However, for π monomial or unitarizable it follows from parts (5) and (6) of Theorem A respectively.
- (iii) We prove analogs of items (1), (3), and (4) of the theorem also for the category of Harish-Chandra modules.

1.1. Related results.

As was mentioned earlier, the non-archimedean counterpart of this paper was done in [BZ77]. In the archimedean case, an analogous notion to the notion of highest derivative was introduced for irreducible unitary representations in [Sah89] and called “adduced representation”.

The case of smooth representations over archimedean fields differs from the above cases in several ways. First of all, we do not have a suitable category of representations of P_n . The existence of such a category in other cases was crucial for the study of derivatives.

Another difference is the relation between the derivative and the classification of irreducible representations. In the non-archimedean case, the theory of derivatives was the base for the Zelevinsky classification. In the unitary case, the notion of adduced representation is closely related with the Vogan classification (see §§1.3.2 and §§4.1.3).

In our case, we do not currently have a classification that is suitable for the theory of derivatives. The Langlands classification is not compatible with the notion of derivative. In particular, it is hard to read from the Langlands classification the annihilator variety or even the depth of the representation, which are crucial notions in the study of derivatives. We hope that eventually it will be possible to make an archimedean analog of the Zelevinsky classification. However, it seems to be quite difficult. Let us explain why.

The Langlands classification presents any irreducible representation as a “smallest” subquotient of a parabolic induction of a discrete series representation. In the non-archimedean case the discrete series representations can be presented as “largest” subquotients of parabolic induction of cuspidal representations. The Zelevinsky classification is dual (under the Zelevinsky involution) to the Langlands classification. Namely, the Zelevinsky classification presents any irreducible representation as a “largest” subquotient of a generalized Speh representation corresponding to a segment of cuspidal representations and any such Speh representation as a “smallest” subquotient of a parabolic induction of a cuspidal representation.

Such a nice picture cannot exist in the archimedean case or even in the complex case. Indeed, in this case only G_1 has discrete series representations. Thus one would expect that the natural analog of generalized Speh representation as above exists only for G_1 . Thus, the naive analogy would suggest that any irreducible representation is the “largest” subquotient of a principal series representation. This is not true. Moreover, it is even not true that any irreducible representation is the “largest” subquotient of a monomial representation (i.e. a Bernstein-Zelevinsky product of characters), or even the “largest” subquotient of a BZ-product of finite-dimensional representations.

The n -th derivative of representations of G_n is the Whittaker functor. Thus, a special case of Theorem A implies that the Whittaker functor is exact and maps a principal series representation to a one-dimensional space. This is known for any quasi-split reductive group by [Kos78] and [CHM00].

1.2. Structure of our proof.

We start working in the Harish-Chandra category. We show that for a Harish-Chandra module π of depth d , $D^d(\pi)$ is an admissible Harish-Chandra module over G_{n-d} . From this we deduce that $D^d(\pi) = \Phi^{d-1}(\pi)$ and $D^k(\pi) = 0$ for any $k > d$.

Then we analyze the functor Φ^k as a functor from $\mathcal{M}_\infty(G_n)$ to the category of representations of \mathfrak{p}_{n-k} . We prove that it is exact and for any $\pi \in \mathcal{M}_\infty(G_n)$, $\Phi^k(\pi)$ is a Hausdorff space. This means that $\mathfrak{u}_n^k(\pi \otimes \psi^k)$ is a closed subspace. In fact, we prove those statements for a wider class of representations of \mathfrak{p}_n .

Then we deduce items (1)-(4) of Theorem A from the above results.

We prove (5) by computing Φ^{d-1} on certain representations of \mathfrak{p}_n using the results on exactness and Hausdorffness of Φ for those representations.

Finally, we prove (6) using (1)-(5), [GS], and the Vogan classification.

1.2.1. Admissibility. Let \mathfrak{n}_n denote the Lie algebra of upper triangular nilpotent $n \times n$ matrices. A Harish-Chandra module π is admissible if and only if it is finitely generated over \mathfrak{n}_n (see Theorem 2.2.2). Thus, we know that $\Phi^{d-1}(\pi)$ is finitely generated over \mathfrak{n}_{n-d+1} and we need to show that it is in fact finitely generated over \mathfrak{n}_{n-d} .

To do that we use two invariants of modules over Lie algebras: annihilator variety (see 2.3) and associated variety (see 5.2). Both are analogs of the notion of support of a module over a commutative algebra. Both are subvarieties of the dual space to the Lie algebra, and the annihilator variety includes the associated variety. The definition of the associated variety requires the module to be filtered, but the resulting variety does not depend on the choice of a good filtration on the module.

To prove that $\Phi^{d-1}(\pi)$ is finitely generated over \mathfrak{n}_{n-d} we show that the associated variety of $\Phi^{d-1}(\pi)$, viewed as a module over \mathfrak{n}_{n-d+1} , is included in \mathfrak{n}_{n-d}^* . Using a lemma that we prove in §§5.5, we deduce this from the bound on the annihilator variety of π that we have by definition of the depth of π .

1.2.2. Exactness and Hausdorffness. It turns out to be convenient to prove the exactness and the Hausdorffness together. Let us describe the proof of exactness, and the proof of Hausdorffness will be incorporated into this argument.

The ideas of the proof come from two sources: the proof of the case $k = n$ given in [CHM00] and the analogy with the p -adic case, where one can use the language of equivariant l -sheaves.

In [CHM00], the first step of the proof is reduction to the statement that the principal series representations are acyclic, using the Casselman embedding theorem. This step works in our case as well. Then in [CHM00] they prove that the principal series representations are acyclic by decomposing them with respect to the decomposition of the flag variety into U_n^n orbits. In their case, there is a finite number of U_n^n orbits and each orbit corresponds to an acyclic representation. In our case, U_n^k has infinitely many orbits on the flag variety and some of them give rise to non-acyclic representations.

Eventually we show that these problematic orbits do not contribute to the higher $(\mathfrak{u}_n^k, \psi_n^k)$ -cohomologies of principal series since they are immersed in smooth families of orbits most of which give rise to acyclic representations. It is difficult to see that directly, therefore we turn to the strategy inspired by equivariant sheaves.

If we had a nice category of P_n -representations, which like in the p -adic case was equivalent to a nice category of G_{n-1} -equivariant sheaves on V_n^* , then the functor Φ would be a composition of a restriction to an open set and an equivalence of categories, and thus it should be exact. Unfortunately it seems too hard at this point to define such categories. Rather than doing this we define a certain class of P_n -representations. We define it using specific examples and constructions and not by demanding certain properties. We do not view this class as a “nice category of representation of P_n ”, in particular we do not analyze it using equivariant sheaves. However it is sufficient for the proof of exactness. Namely we prove that this class includes the principal series representations of G_n . We prove that it is closed under Φ . Finally, we prove that its objects are acyclic with respect to Φ .

In order to do this we develop some tools for computing the homology of representations constructed in a geometric way.

1.2.3. *Depth derivative of monomial representations.* In the non-archimedean case, [BZ77] provides a description of any derivative of a Bernstein-Zelevinsky product $\rho_1 \times \rho_2$ in terms of derivatives of ρ_1 and ρ_2 . In particular, this description says that the highest derivative of a product is the product of highest derivatives. This immediately implies the natural analog of Theorem A (5). This description is proved using geometric analysis of the Bernstein-Zelevinsky product as the space of sections of the corresponding sheaf on the corresponding Grassmannian.

This method is hard to translate to the archimedean case for the following reasons.

- The analogous statement for the derivative as we defined it here does not hold. If we would not replace the functor of coinvariants by the functor of generalized coinvariants then the analogous statement might hold. However, we would lose the exactness, which played a crucial role in the proof in [BZ77].
- We do not have an appropriate language of infinite-dimensional sheaves.

For these reasons, we prove a product formula only for the depth derivatives, and only for products of characters. The drawback of that approach is that we have to consider product of more than two characters.

We prove the product formula using the geometric description of monomial representations. More specifically, we decompose the corresponding flag variety into P_n -orbits. This decomposition gives a filtration of the monomial representation. We compute the functor Φ on each of the associated graded pieces in geometric terms, and use the exactness of Φ to proceed by induction.

1.3. Applications.

1.3.1. *Degenerate Whittaker models.* Let $N_n < G_n$ be the nilradical of a Borel subgroup. Let χ be any unitary character of N_n . A (degenerate) Whittaker functional on a smooth representation of G_n is an (N_n, χ) -equivariant functional. Such functionals were studied in [GS]. In particular, [GS] associates a character χ_π of N_n to any irreducible representation π , using the annihilator variety of π and proves existence of the corresponding degenerate Whittaker functionals for unitarizable π . In this paper we deduce uniqueness of those functionals from Theorem A.

1.3.2. *Adduced representations.* In [Sah89, Sah90, SaSt90] adduced representations were computed for characters, Stein complementary series and Speh representations. Moreover it was proven that adduced representation of a Bernstein-Zelevinsky product is the product of the corresponding adduced representations. Thus, by Vogan classification, the task of computing adduced representations was reduced to the case of Speh complementary series. In this paper we perform this computation, based on Theorem A.

1.3.3. *Future applications.* In [CPS], Cogdell and Piatetski-Shapiro use Bernstein-Zelevinsky derivatives to compute local Rankin-Selberg integrals for $GL_n \times GL_m$ over p-adic fields. Hopefully, the derivatives defined in this paper can be used to compute Rankin-Selberg integrals at the archimedean place in a similar way. J. Cogdell has informed us that this is being investigated by his student J. Chai in his PhD thesis at Ohio State University.

1.4. Tools developed in this paper.

1.4.1. *A bound on annihilator variety in terms of coinvariants.* In order to prove the admissibility of the depth's derivative we need to show a certain bound on its associated variety. For this we analyze the following situation: Let $\mathfrak{h}_1 \triangleleft \mathfrak{h}_2 < \mathfrak{g}$ be Lie algebras. Let ψ be a character of \mathfrak{h}_1 that is stabilized by \mathfrak{h}_2 and let π be a filtered \mathfrak{g} -module. We have certain bounds on the annihilator variety of π in \mathfrak{g}^* and we are interested in the associated variety of $\pi_{\mathfrak{h}_2, \psi}$ in $(\mathfrak{h}_1/\mathfrak{h}_2)^*$.

In §§5.5 we provide some bounds on this associated variety under certain technical assumptions.

1.4.2. *The relative Shapiro lemma.* In the proof of the exactness of the derivatives we are interested in the study of homology of representations of the following type. Let a real algebraic group G act on a real algebraic manifold X , and let \mathcal{E} be a G -equivariant bundle over X . Consider the space $\mathcal{S}(X, \mathcal{E})$ of Schwartz sections of this bundle (see §6). This is a representation of G . For technical reasons we prefer to study Lie algebra homology rather than Lie group homology. Thus we will impose some homotopic assumptions on G and X so that there will be no difference. In the case when X is homogeneous, i.e. $X = G/H$, a version of the Shapiro Lemma states, under suitable homotopic and unimodularity assumptions on G and H , that $H_*(\mathfrak{g}, \mathcal{S}(X, \mathcal{E})) = H_*(\mathfrak{h}, \mathcal{E}_{[1]})$, where $\mathcal{E}_{[1]}$ denotes the fiber of \mathcal{E} at the class of 1 in G/H (cf. [AG10]).

In Appendix A we prove a generalization of this statement to the relative case. Namely, let G act on $X \times Y$ and \mathcal{E} be a G -equivariant bundle over $X \times Y$. Suppose that $X = G/H$. Then, under the same assumptions on G and H , we have

$$H_*(\mathfrak{g}, \mathcal{S}(X \times Y, \mathcal{E})) = H_*(\mathfrak{h}, \mathcal{S}([1] \times Y, \mathcal{E}_{[1] \times Y})).$$

For the zero homology this statement is in fact Frobenius descent (cf. [AG09, Appendix B.3]).

1.4.3. *Homology of families of characters.* In the proof of the exactness of the derivatives, we use the relative Shapiro lemma to reduce to the study of $H_*(\mathfrak{g}, \mathcal{S}(X, \mathcal{E}))$, where the action of G on X is trivial. Moreover, in our case \mathcal{E} is a line bundle and \mathfrak{g} is abelian. In §10 we prove that $\mathcal{S}(X, \mathcal{E})$ is acyclic and compute $H_0(\mathfrak{g}, \mathcal{S}(X, \mathcal{E}))$ under certain conditions on the action of G on \mathcal{E} .

1.4.4. *Tempered equivariant bundles.* Since we discuss non-algebraic (e.g. unitary) characters, it might become necessary to consider non-algebraic geometric objects. We tried to avoid that, but had to consider vector bundles of algebraic nature with a non-algebraic action of a group. A similar difficulty arose in [AG], and the notion of tempered equivariant bundle was defined in order to deal with it. In this paper we develop more tools to work with such bundles.

1.5. Structure of the paper.

In §2 we give the necessary preliminaries on Harish-Chandra modules, admissible smooth representations, annihilator varieties and Bernstein-Zelevinsky product.

In §3 we formulate the main results and prove some implications between them. We reduce Theorem A to three statements: admissibility of the depth derivative in the Harish-Chandra category, exactness and Hausdorffness of Φ^k in the smooth category, and computation of the depth derivative of monomial representations in the smooth category.

In §4 we discuss the relation of the notion of depth derivative to the notion of adduced representation and deduce the applications discussed in §§1.3. We also provide the necessary preliminaries on the Vogan classification, adduced representations and degenerate Whittaker models.

In §5 we prove admissibility of the depth derivative in the Harish-Chandra category. This section is purely algebraic. In 5.1 we give an overview of the proof. In §§5.2 we give some preliminaries on filtrations and associated varieties. In §§5.3-5.5 we formulate and prove a result (the algebraic key lemma) that connects the annihilator variety of a representation π to the associated variety of co-equivariants of π with respect to a certain subalgebra and its character. In §§5.6 we deduce the admissibility from the algebraic key lemma.

The proof of exactness and Hausdorffness of Φ^k and the computation of the depth derivative of the monomial representations are based on the theory of Schwartz functions on Nash manifolds. In §6 we review this theory, including the notion of equivariant tempered bundle.

In §7 we prove that Φ^k is exact functor on the category of smooth admissible representations and that $\Phi^k(\pi)$ is Hausdorff for any reducible smooth admissible representation π . In §§7.1 we introduce the class of good representations of \mathfrak{p}_n . The construction of this class is based on representations with simple geometric descriptions that we call geometric representations. We prove that this class includes the principal series representations of G_n , is closed under Φ and that its objects are acyclic with respect to Φ . In order to prove the last two statements we use Lemma 7.1.1 (the analytic key lemma), which is proved in §10.

In §8 we compute the depth derivative of a monomial representation. This computation is based on the computation of certain geometric representations. We formulate the results of this computation in §§8.4, and prove them in §§9.9.

In §9 we give the postponed proofs of the technical lemmas that were used in sections 3, 7, and 8. For this we need some preliminaries on homological algebra, topological linear spaces and Schwartz functions on Nash manifolds, which we give in subsections 9.2, 9.4, and 9.6.

In §10 we prove the analytic key lemma (Lemma 7.1.1).

In Appendix A we prove a version of the Shapiro lemma which is crucial for the proof of Lemma 7.1.1.

In Appendix B we present a proof of Proposition 5.2.7 which is an important tool for using filtrations on \mathfrak{g}_n -modules. This proposition was proven by O. Gabber and written in the unpublished lecture notes [Jos80]. We present a proof here for the sake of completeness.

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2. PRELIMINARIES

2.1. Notation and conventions.

- We will denote real algebraic groups by capital Latin letters and their complexified Lie algebras by small Gothic letters.
- Let \mathfrak{g} be a complex Lie algebra. We denote by $\mathcal{M}(\mathfrak{g})$ the category of (arbitrary) \mathfrak{g} -modules. Let ψ be a character of \mathfrak{g} . For a module $M \in \mathcal{M}(\mathfrak{g})$ denote by $M^{\mathfrak{g}}$ the space of \mathfrak{g} -invariants, by $M^{\mathfrak{g}, \psi}$ the space of (\mathfrak{g}, ψ) -equivariants, by $M_{\mathfrak{g}}$ the space of coinvariants, i.e. $M_{\mathfrak{g}} := M/\mathfrak{g}M$ and by $M_{\mathfrak{g}, \psi}$ the space of (\mathfrak{g}, ψ) -coequivariants, i.e. $M_{\mathfrak{g}, \psi} := (M \otimes (-\psi))_{\mathfrak{g}}$.
- We also denote by $M_{gen, \mathfrak{g}}$ the space of the generalized co-invariants, i.e.

$$M_{gen, \mathfrak{g}} = \varprojlim_l M/\text{Span}(\{\alpha v \mid v \in M, \alpha \in (\mathfrak{g})^{\otimes l}\}).$$

- By a composition of n we mean a tuple (n_1, \dots, n_k) of natural numbers such that $n_1 + \dots + n_k = n$. By a partition we mean a composition which is non-increasing, i.e. $n_1 \geq \dots \geq n_k$.
- For a composition $\lambda = (n_1, \dots, n_k)$ of n we denote by P_{λ} the corresponding block-upper triangular parabolic subgroup of G_n . For example, $P_{(1, \dots, 1)} = B_n$ denotes the standard Borel subgroup, $P_{(n)} = G_n$ and $P_{(n-1, 1)}$ denotes the standard maximal parabolic subgroup that includes P_n .

2.2. Harish-Chandra modules and smooth representations. In this subsection we fix a real reductive group G , a minimal parabolic subgroup of $P \subset G$, and let N denote the nilradical of P . We also fix a maximal compact subgroup $K \subset G$. Let $\mathfrak{g}, \mathfrak{n}, \mathfrak{k}$ denote the complexified Lie algebras of G, N, K , and let $Z_G := U(\mathfrak{g})^G$.

Definition 2.2.1. A (\mathfrak{g}, K) -module is a \mathfrak{g} -module π with a locally finite action of K such the two induced actions of \mathfrak{k} coincide and $\pi(\text{ad}(k)(X)) = \pi(k)\pi(X)\pi(k^{-1})$ for any $k \in K$ and $X \in \mathfrak{g}$.

A finitely-generated (\mathfrak{g}, K) -module is called admissible if any representation of K appears in it with finite (or zero) multiplicity.

Theorem 2.2.2 (Harish-Chandra, Osborne, Stafford, Wallach). *Let π be a finitely generated (\mathfrak{g}, K) -module. Then the following properties of π are equivalent.*

- (1) π is admissible.
- (2) π has finite length.
- (3) π is Z_G -finite.
- (4) π is finitely generated over \mathfrak{n} .

The implications $(1) \Rightarrow (2)$ and $(4) \Rightarrow (1)$ are proven using the Casselman-Jacquet functor, see [Wall88, §§4.2]. The implication $(2) \Rightarrow (3)$ follows from Schur's Lemma, and the implication $(3) \Rightarrow (4)$ is proven in [Wall88, §§3.7]

Notation 2.2.3. For a real reductive group G we denote by \mathfrak{g} its complexified Lie algebra, by $\mathcal{M}(\mathfrak{g})$ the category of \mathfrak{g} -modules, by $\mathcal{M}_\infty(G)$ the category of smooth admissible Fréchet representations of G of moderate growth and by $\mathcal{M}_{HC}(G)$ the category of admissible Harish-Chandra modules. Note that both $\mathcal{M}_\infty(G)$ and $\mathcal{M}_{HC}(G)$ are naturally subcategories of $\mathcal{M}(\mathfrak{g})$. We denote by $HC : \mathcal{M}_\infty(G) \rightarrow \mathcal{M}_{HC}(G)$ the functor of K -finite vectors.

Theorem 2.2.4 (Casselman-Wallach, see [Wall92], §§11.6.8). *The functor $HC : \mathcal{M}_\infty(G) \rightarrow \mathcal{M}_{HC}(G)$ is an equivalence of categories.*

In fact, Casselman and Wallach construct an inverse functor $\Gamma : \mathcal{M}_{HC}(G) \rightarrow \mathcal{M}_\infty(G)$, that is called Casselman-Wallach globalization functor (see [Wall92, Chapter 11] or [Cas89] or, for a different approach, [BK]).

Corollary 2.2.5.

- (i) *The category $\mathcal{M}_\infty(G)$ is abelian.*
- (ii) *Any morphism in $\mathcal{M}_\infty(G)$ has closed image.*

Proof. (i) The category $\mathcal{M}_{HC}(G)$ is clearly abelian and by the theorem it is equivalent to $\mathcal{M}_\infty(G)$.

(ii) Let $\phi : \pi \rightarrow \tau$ be a morphism in $\mathcal{M}_\infty(G)$. Let $\tau' = \overline{\text{Im } \phi}$, $\pi' = \pi / \ker \phi$ and $\phi' : \pi' \rightarrow \tau'$ be the natural morphism. Clearly ϕ' is monomorphic and epimorphic in the category $\mathcal{M}_\infty(G)$. Thus by (i) it is an isomorphism. On the other hand, $\text{Im } \phi' = \text{Im } \phi \subset \overline{\text{Im } \phi} = \tau'$. Thus $\text{Im } \phi = \overline{\text{Im } \phi}$. \square

Theorem 2.2.6 (Casselman subrepresentation theorem, see [CM82], Proposition 8.23). *Let π be a finitely generated admissible (\mathfrak{g}, K) -module and P be a minimal parabolic subgroup of G . Then there exists a finite-dimensional representation σ of P such that π may be imbedded into $\text{Ind}_P^G(\sigma)$.*

2.3. The annihilator variety and associated partition. For an associative algebra A the annihilator of a module (τ, W) is

$$\text{Ann}(\tau) = \{a \in A : \tau(a)w = 0 \text{ for all } w \in W\}.$$

If A is abelian then the support of τ is defined to be the variety corresponding to the ideal $\text{Ann}(\tau)$, i.e. $\text{Zeroes}(\text{Ann}(\tau))$.

If (τ, W) is a module for a Lie algebra \mathfrak{g} , then one can apply the above considerations to the enveloping algebra $U(\mathfrak{g})$. While $U(\mathfrak{g})$ is not abelian it admits a natural filtration U^n such that $gr(U(\mathfrak{g}))$ is the symmetric algebra $\text{Sym}(\mathfrak{g})$, and hence one has a symbol map σ from $U(\mathfrak{g})$ to $\text{Sym}(\mathfrak{g})$. We let $gr(\text{Ann}(\tau))$ be the ideal in $\text{Sym}(\mathfrak{g})$ generated by the symbols $\{\sigma(a) \mid a \in \text{Ann}(\tau)\}$ and define the annihilator variety of τ to be

$$\mathcal{V}(\tau) = \text{Zeroes}(gr(\text{Ann}(\tau))) \subset \mathfrak{g}^*$$

If \mathfrak{g} is a complex reductive Lie algebra and M is an irreducible \mathfrak{g} -module, then it was shown by Borho-Brylinski (see [BB82]) and Joseph (see [Jos85]) that $\mathcal{V}(M)$ is the closure $\overline{\mathcal{O}}$ of a single nilpotent coadjoint orbit \mathcal{O} , that we call the associated orbit of M . If G is a reductive group and $\pi \in \mathcal{M}_\infty(G)$ is admissible then π^{HC} is dense in π and hence $\mathcal{V}(\pi) = \mathcal{V}(\pi^{HC})$.

2.3.1. The case of G_n . Suppose first that $F = \mathbb{R}$. Then $\mathfrak{g}_n = \mathfrak{gl}(n, \mathbb{C})$ and we identify \mathfrak{g}_n and \mathfrak{g}_n^* with the space $n \times n$ complex matrices, in the usual manner. By Jordan's theorem, nilpotent orbits in \mathfrak{g}_n^* are given by partitions of n , i.e. tuples (n_1, \dots, n_k) such that $n_1 \geq \dots \geq n_k$ and $n_1 + \dots + n_k = n$. For a partition λ we denote by \mathcal{O}_λ the corresponding nilpotent coadjoint orbit. We sometimes use exponential notation for partitions; thus $4^2 2^1 1^3$ denotes $(4, 4, 2, 1, 1, 1)$.

If $\pi \in \mathcal{M}_\infty(G_n)$ is irreducible and λ is the partition of n such that $\mathcal{V}(\pi) = \overline{\mathcal{O}_\lambda}$ we call λ the *associated partition* of π and denote $\lambda = AP(\pi)$. For example, if π is finite-dimensional then $\mathcal{V}(\pi) = \{0\}$ and $AP(\pi) = 1^n$ and if π is generic then, by [Kos78], $\mathcal{V}(\pi)$ is the nilpotent cone of \mathfrak{g}_n^* and $AP(\pi) = n^1$.

Let us introduce the following definition of depth.

Definition 2.3.1. Let $\pi \in \mathcal{M}_{HC}(G_n)$. Define $\text{depth}(\pi)$ to be the smallest number d such that $A^d = 0$ for any $A \in \mathcal{V}(\pi)$.

It is easy to see that for an irreducible representation π with associated partition (n_1, \dots, n_k) , $\text{depth}(\pi) = n_1$ and the depth of an extension of two representations is the maximum of their depths.

Let us now consider the case $F = \mathbb{C}$. Then $\mathfrak{g}_n = \mathfrak{gl}(n, \mathbb{C}) \oplus \mathfrak{gl}(n, \mathbb{C})$ and coadjoint nilpotent orbits are given by pairs of partitions. However, $\pi \in \mathcal{M}_\infty(G_n)$ is irreducible then the maximal orbit in $\mathcal{V}(\pi)$ is symmetric and thus corresponds to a single partition that we call the associated partition. For any $\pi \in \mathcal{M}_{HC}(GL_n(\mathbb{C}))$ we define $\text{depth}(\pi)$ to be the smallest number d such that $A^d = B^d = 0$ for any $(A, B) \in \mathcal{V}(\pi)$.

2.4. Parabolic induction and Bernstein-Zelevinsky product. Let G be a real reductive group, P be a parabolic subgroup, M be the Levi quotient of P and $pr : P \rightarrow M$ denote the natural map.

Notation 2.4.1.

- For a Lie group H we denote by Δ_H the modulus character of H , i.e. the absolute value of the determinant of the infinitesimal adjoint action.
- For $\pi \in \mathcal{M}_\infty(M)$ we denote by $I_M^G(\pi)$ the normalized parabolic induction of π , i.e. the space of smooth functions $f : G \rightarrow \pi$ such that $f(pg) = \Delta_H(p)^{1/2} \Delta_G(p)^{-1/2} \pi(pr(p))f(g)$, with the action of G given by $(I_M^G(\pi)(g)f)(x) := f(xg)$.

The behavior of annihilator variety under parabolic induction is described by the following theorem.

Theorem 2.4.2. Note that we have a natural embedding $\mathfrak{m}^* \hookrightarrow \mathfrak{p}^*$ and a natural projection $r : \mathfrak{g}^* \rightarrow \mathfrak{p}^*$. Let $\pi \in \mathcal{M}_\infty(M)$. Then $\mathcal{V}(I_M^G(\pi)) = G_{\mathbb{C}} \cdot r^{-1}(\mathcal{V}(\pi))$.

This theorem is well-known and can be deduced from [BB89, Theorem 2].

2.4.1. Bernstein-Zelevinsky product. We now introduce the Bernstein-Zelevinsky product notation for parabolic induction.

Definition 2.4.3. If $\alpha = (n_1, \dots, n_k)$ is a composition of n and $\pi_i \in \mathcal{M}_\infty(G_{n_i})$ then $\pi_1 \otimes \dots \otimes \pi_k$ is a representation of $L_\alpha \approx G_{\alpha_1} \times \dots \times G_{\alpha_k}$. We define

$$\pi_1 \times \dots \times \pi_k = I_{P_\alpha}^{G_n}(\pi_1 \otimes \dots \otimes \pi_k)$$

$\pi_1 \times \dots \times \pi_k$ will be referred to below as the Bernstein-Zelevinsky product, or the BZ-product, or sometimes just the product of π_1, \dots, π_k . It is well known (see e.g. [Wall92, §§12.1]) that the product is commutative in the Grothendieck group. From Theorem 2.4.2 we obtain

Corollary 2.4.4. Let $\pi_1 \in G_{n_1}$ and $\pi_2 \in G_{n_2}$. Then $\text{depth}(\pi_1 \times \pi_2) = \text{depth}(\pi_1) + \text{depth}(\pi_2)$.

3. MAIN RESULTS

Notation 3.0.1.

- Fix F to be either \mathbb{R} or \mathbb{C} .
- $G_n := GL(n, F)$, we consider the standard embeddings $G_n \subset G_m$ for any $m > n$, we denote the union by G_∞ and all the groups we will consider will be embedded to G_∞ in a standard way.
- We denote by $P_n \subset G_n$ the mirabolic subgroup (consisting of matrices with last row $(0, \dots, 0, 1)$).
- Let $V_n \subset P_n$ be the last column. Note that $V_n \cong F^{n-1}$ and $P_n = G_{n-1} \ltimes V_n$. Let $U_n^k := V_{n-k+1} V_{n-k+2} \cdots V_n$ and $S_n^k := G_{n-k} U_n^k$. Note that U_n^k is the unipotent radical of S_n^k . Let $N_n := U_n^k$.
- Fix a non-trivial unitary additive character θ of F , given by $\theta(x) = \exp(\sqrt{-1}\pi \text{Re } x)$.
- Let $\bar{\psi}_n^k : U_n^k \rightarrow F$ be the standard non-degenerate homomorphism, given by $\bar{\psi}_n^k(u) = \sum_{j=n-k}^{n-1} u_{j,j+1}$ and let $\psi_n^k := \theta \circ \bar{\psi}_n^k$.

We will usually omit the n from the notations U_n^k and S_n^k , and both indexes from ψ_n^k .

Definition 3.0.2. Define functors $\Phi : \mathcal{M}(\mathfrak{p}_n) \rightarrow \mathcal{M}(\mathfrak{p}_{n-1})$ by $\Phi(\pi) := \pi_{\mathfrak{v}_n, \psi} \otimes |\det|^{-1/2}$ and $\Psi, \Psi_0 : \mathcal{M}(\mathfrak{p}_n) \rightarrow \mathcal{M}(\mathfrak{g}_{n-1})$ by $\Psi(\pi) := \pi_{gen, \mathfrak{v}_n}$ and $\Psi_0(\pi) := \pi_{\mathfrak{v}_n}$.

For a \mathfrak{p}_n -module π we define three notions of derivative:

- (1) $E^k(\pi) := \Phi^{k-1}(\pi) := \pi_{\mathfrak{u}^{k-1}, \psi^{k-1}} \otimes |\det|^{-(k-1)/2}$. Clearly it has a structure of a \mathfrak{p}_{n-k+1} - representation.
- (2) $D^k(\pi) := \Psi(E^k(\pi))$.
- (3) $B^k(\pi) := \Psi_0(E^k(\pi))$.

Note that the derivative functor D^k was defined in the introduction. For convenience we will also use untwisted versions of the above functors, defined by $\tilde{\Phi}(\pi) := \Phi(\pi) \otimes |\det|^{1/2}$, and $\tilde{E}^k(\pi) := E^k(\pi) \otimes |\det|^{(k-1)/2}$.

We can restrict those functors both to the smooth and the Harish-Chandra category and get B_∞^k , D_∞^k , E_∞^k , B_{HC}^k , D_{HC}^k and E_{HC}^k . Note that if $\pi \in \mathcal{M}_\infty(G_n)$ then $D_\infty^k(\pi)$ has a natural structure of a P_{n-k+1} topological representation and if $\pi \in \mathcal{M}_{HC}(G_n)$ the $D_{HC}^k(\pi)$ has a natural structure of a K' representation where K' is the maximal compact subgroup of G_{n-k} . The same is true for the functors B and E .

We have natural maps: $E^k \rightarrow D^k \rightarrow B^k$, $HC \circ B_\infty^k \rightarrow B_{HC}^k \circ HC$, $HC \circ D_\infty^k \rightarrow D_{HC}^k \circ HC$ and $HC \circ E_\infty^k \rightarrow E_{HC}^k \circ HC$. Here HC is the functor of taking K -finite vectors and the last three maps are maps of K representations and \mathfrak{p} representations. However note that the objects are not necessary (\mathfrak{g}, K) -modules since they do not need to be Hausdorff.

Proposition 3.0.3. Let $\pi \in \mathcal{M}_{HC}(G_n)$. Then $B_{HC}^k(\pi)$ is admissible for any $1 \leq k \leq n$.

Proof. By Theorem 2.2.2, π is finitely generated over \mathfrak{n}_n . Thus, $B_{HC}^k(\pi)$ is finitely generated over \mathfrak{n}_{n-k} and thus, by Theorem 2.2.2 again, $B_{HC}^k(\pi)$ is admissible. \square

Remark 3.0.4. Let $\pi \in \mathcal{M}_{HC}(G_n)$, and let $S' \in \mathbb{C}^n$ be the multiset corresponding to an infinitesimal character of $B^k(\pi)$. Then S' is obtained from the multiset corresponding to some infinitesimal character of π by deleting k of the elements and adding $1/2$ to each of the remaining ones. This is proven by the argument in the proof of [GS, Proposition 5.4.4].

In the §5 we prove the following theorem

Theorem 3.0.5. Let $\pi \in \mathcal{M}_{HC}^d(G_n)$. Then the restriction of $E_{HC}^d(\pi)$ to \mathfrak{g}_{n-d} is admissible.

Corollary 3.0.6. Let $\pi \in \mathcal{M}_{HC}^d(G_n)$. Then \mathfrak{v}_{n-d+1} acts nilpotently on $E_{HC}^d(\pi)$. Namely, there exists a number k such that for any $X \in \mathfrak{v}_{n-d}$, X^k acts by zero on $E_{HC}^d(\pi)$.

Proof. Let $\tau := E_{HC}^d(\pi)$. Since it is admissible over \mathfrak{g}_{n-d} , it is finite over the center of $U(\mathfrak{g}_{n-d})$. Hence there exists a polynomial p such that $\tau(p(I)) = 0$, where $I \in \mathfrak{g}_{n-d}$ denotes the identity matrix. Let k be the degree of p and $X \in \mathfrak{v}_{n-d}$ be any element. We will show that $\tau(X)^k = 0$.

Note that $[I, X] = X$ and hence $ad(X)^k(I^k) = k!(-X)^k$ and $ad(X)^k I^{k-i} = 0$ for any $i > 0$. Thus $ad(X)^k(p(I))$ is proportional to X^k . On the other hand, $\tau(p(I)) = 0$, hence $\tau(ad(X)^k(p(I))) = 0$ and thus $\tau(X)^k = 0$. \square

Corollary 3.0.7. Let $\pi \in \mathcal{M}_{HC}^d(G_n)$. Then

- (1) $D_{HC}^d(\pi) = E_{HC}^d(\pi)$.
- (2) $E_{HC}^{d+1}(\pi) = D_{HC}^{d+1}(\pi) = B_{HC}^{d+1}(\pi) = 0$.

Theorem 3.0.8. For any $0 < k \leq n$

- (1) E_∞^k is an exact functor
- (2) For any $\pi \in \mathcal{M}_\infty(G_n)$, the natural (quotient) topology on $E_\infty^k(\pi)$ is Hausdorff, i.e. $\mathfrak{u}^k(\pi \otimes (-\psi^k))$ is closed in π .

For proof see §7.

Corollary 3.0.9. Let $\pi \in \mathcal{M}_\infty(G_n)$ be of depth d . Let $0 < k \leq n$. Then

- (1) The natural map $p : E_{HC}^k(HC(\pi)) \rightarrow HC(E_\infty^k(\pi))$ is onto.
- (2) $E_\infty^d(\pi) = D_\infty^d(\pi) \in \mathcal{M}_\infty(G_{n-d})$.

$$(3) \quad E_{\infty}^{d+1}(\pi) = D_{\infty}^{d+1}(\pi) = B_{\infty}^{d+1}(\pi) = 0.$$

For the proof we will need the following standard lemma.

Lemma 3.0.10. *If a locally convex Hausdorff topological vector space W has a dense finite dimensional subspace then W is finite dimensional.*

For the convenience of the reader, we include the proof in §§9.1.

Proof of Corollary 3.0.9. Let us first prove part (1). The quotient map $\pi \rightarrow E_{\infty}^k(\pi)$ is onto. Thus, $HC(\pi) \rightarrow E_{\infty}^k(\pi)$ has dense image, hence $E_{HC}^k(HC(\pi)) \rightarrow E_{\infty}^k(\pi)$ has dense image and hence p has dense image. Let ρ be a K_{n-k} -type. Consider $p^{\rho} : (E_{HC}^k(HC(\pi)))^{\rho} \rightarrow (HC(E_{\infty}^k(\pi)))^{\rho}$. It must also have dense image. By Theorem 3.0.5, $(E_{HC}^k(HC(\pi)))^{\rho}$ is finite dimensional and by Theorem 3.0.8, $(HC(E_{\infty}^k(\pi)))^{\rho}$ is Hausdorff. Thus, p^{ρ} is onto for any ρ and hence, by Lemma 3.0.10, p is onto. Thus (1) holds.

By Theorem 3.0.5, $E_{HC}^d(HC(\pi))$ is admissible. Thus $HC(E_{\infty}^d(\pi))$ is admissible and thus $E_{\infty}^d(\pi) \in \mathcal{M}_{\infty}(G_{n-d})$. By Corollary 3.0.6, \mathfrak{v}_{n-d+1} acts nilpotently on $E_{HC}^d(HC(\pi))$, and hence, by (1), on $HC(E_{\infty}^d(\pi))$ and hence, by continuity, \mathfrak{v}_{n-d+1} acts nilpotently on $E_{\infty}^d(\pi)$. This implies (2) and (3). \square

Note that this does not prove that the d -th derivative is non-zero. However it is true; see the remarks following Theorem A in the introduction.

Corollary 3.0.11. *Let $\pi \in \mathcal{M}_{\infty}(G_n)$ be of depth d . Then $B_{\infty}^d(\pi) \in \mathcal{M}_{\infty}(G_{n-d})$.*

Proof. By Corollary 3.0.9, $E_{\infty}^d(\pi) \in \mathcal{M}_{\infty}(G_{n-d})$. Note that $B_{\infty}^d(\pi)$ is the cokernel of the action map $a : \mathfrak{v}_{n-d+1} \otimes E_{\infty}^d(\pi) \rightarrow E_{\infty}^d(\pi)$. Thus, it is enough to show that $\text{Im}(a)$ is closed. This follows from Corollary 2.2.5. \square

Lemma 3.0.12. *For any $\pi \in \mathcal{M}_{HC}^d(G_n)$, $\mathcal{V}(B^d(\pi)) = \mathcal{V}(E^d(\pi))$.*

Proof. By Corollary 3.0.6, there exists k such that $\mathfrak{v}_{d+1}^k E^d(\pi) = 0$. Consider the descending filtration $F^i(E^d(\pi)) := \mathfrak{v}_{d+1}^i E^d(\pi)$. Then $Gr^0(E^d(\pi)) = B^d(\pi)$, $Gr^i(\pi) \in \mathcal{M}_{HC}(G_{n-d})$ and we have a natural morphism $Gr^i(\pi) \otimes \mathfrak{v}_{d+1} \rightarrow Gr^{i+1}(\pi)$, where we view \mathfrak{v}_{d+1} as the standard representation of G_{n-d} . Thus, $\mathcal{V}(E^d(\pi)) = \cup_{i=1}^k \mathcal{V}(Gr^i(E^d(\pi)))$ and $\mathcal{V}(Gr^i(E^d(\pi))) \supset \mathcal{V}(Gr^{i+1}(E^d(\pi)))$. Thus $\mathcal{V}(B^d(\pi)) = \mathcal{V}(Gr^0(E^d(\pi))) = \mathcal{V}(E^d(\pi))$. \square

By Corollary 3.0.9 the same statement holds for $\pi \in \mathcal{M}_{\infty}^d(G_n)$ and the same proof works.

Theorem 3.0.13. *Let $n = n_1 + \cdots + n_k$ and let χ_i be characters of G_{n_i} . Let $\pi = \chi_1 \times \cdots \times \chi_k$ denote the corresponding monomial representation. Then*

$$E_{\infty}^k(\pi) = E_{\infty}^1(\chi_1) \times \cdots \times E_{\infty}^k(\chi_k) = ((\chi_1)|_{G_{n_1-1}} \times \cdots \times (\chi_k)|_{G_{n_k-1}}).$$

We prove this theorem in §8.

Remark 3.0.14.

- (1) Note that $\text{depth}(\pi) = k$ by Corollary 2.4.4.
- (2) In the special case $n = k$ this theorem implies that the space of Whittaker functionals on a principal series representation is one-dimensional. By the same example we see that an analog of Theorem 3.0.13 for E_{HC} does not hold, since the space of Whittaker functionals on the Harish-Chandra module of a principal series representation has dimension $n!$.

From Theorem 3.0.13, Theorem 3.0.8 and Corollary 3.0.9 we obtain

Corollary 3.0.15. *Let $I = \chi_1 \times \cdots \times \chi_k$ be a monomial representation. Let π be any subquotient of I . Then $E_{\infty}^k(\pi) \cong D_{\infty}^k(\pi) \cong B_{\infty}^k(\pi)$.*

Proof. If $\text{depth}(\pi) < k$ we have $E_{\infty}^k(\pi) = D_{\infty}^k(\pi) = B_{\infty}^k(\pi) = 0$. Otherwise, $\text{depth}(\pi) = k$, $E_{\infty}^k(\pi) \cong D_{\infty}^k(\pi)$ and we have to show that $\mathfrak{v}_{n-d+1} E_{\infty}^k(\pi) = 0$. Note that all the subquotients of a monomial representation have the same infinitesimal character. By Theorem 3.0.13, $E_{\infty}^k(I)$ is again a monomial representation and by Theorem 3.0.8 $E_{\infty}^k(\pi)$ is its subquotient. Thus, the identity matrix $Id \in \mathfrak{g}_{n-d}$ acts

on $E_\infty^k(\pi)$ by a scalar, that we denote by a . Now, let $x \in E_\infty^k(\pi)$ and $v \in \mathfrak{v}_{n-d+1}$. Then $avx = Ivx = [I, v]x + vIx = vx + avx = (a+1)vx$. Thus, $vx = 0$ for any v and x and thus $\mathfrak{v}_{n-d+1}E_\infty^k(\pi) = 0$ as required. \square

In §4 we demonstrate several applications of Theorem 3.0.13 to unitary representations.

3.1. Conjectures and questions. We conjecture that the following generalization of Theorem 3.0.13 holds for Bernstein-Zelevinsky product of arbitrary representations.

Conjecture 3.1.1. *Let $n = n_1 + n_2$ and let $\pi_i \in \mathcal{M}_\infty^{d_i}(G_{n_i})$ for $i = 1, 2$. Let $\pi := \pi_1 \times \pi_2$ and $d := d_1 + d_2$, so that $\pi \in \mathcal{M}_\infty^d(G_n)$. Then $E_\infty^d(\pi) = E_\infty^{d_1}(\pi_1) \times E_\infty^{d_2}(\pi_2)$.*

We think that it is possible to prove this conjecture using the same geometric argument as in the proof of Theorem 3.0.13 (see §8). We believe that the main ingredient we miss are suitable notions of tempered Fréchet bundle and the space of its Schwartz sections.

Our paper leaves the following open questions:

- (1) In Proposition 3.0.3 we show that B^k maps admissible Harish-Chandra modules to admissible, for every k . Is the same true on the smooth category $\mathcal{M}_\infty(G_n)$? In other words, is $B^k(\pi)$ Hausdorff for any $\pi \in \mathcal{M}_\infty(G_n)$ and any k ? We think that the answer is yes and hopefully it can be proven using the methods of §7.
- (2) Is the depth B -derivative functor $B^d : \mathcal{M}_\infty^d(G_n) \rightarrow \mathcal{M}_\infty(G_{n-d})$ exact?
- (3) Is D^k exact for any k ?
- (4) Does B^d map irreducible representations of depth d to irreducible ones? This would in particular imply uniqueness of degenerate Whittaker models for all smooth representations (see §4).

4. HIGHEST DERIVATIVES OF UNITARY REPRESENTATIONS AND APPLICATIONS

In this section we prove two results conjectured in [GS]: uniqueness of degenerate Whittaker models and computation of adduced representations for Speh complementary series. We also prove that for irreducible unitary representations, the three notions of highest derivative coincide and extend the notion of adduced representation. We start with preliminaries on Speh representation, Vogan classification, adduced representation, and degenerate Whittaker models.

4.1. Preliminaries.

Notation 4.1.1. *Let $z \in \mathbb{C}$, and let $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ if $F = \mathbb{R}$ and $\varepsilon \in \mathbb{Z}$ if $F = \mathbb{C}$. Denote by $\chi(n, \varepsilon, z)$ the character of G_n given by*

$$x \mapsto \left(\frac{\det x}{|\det x|} \right)^\varepsilon |\det x|^z.$$

This character is unitary if z is imaginary.

4.1.1. Speh representation. We will use the following description of the Speh representation form [BSS90] and [SaSt90].

Proposition 4.1.2. *The Speh representation $\delta(2m, k)$ is a quotient of*

$$\chi(m, \varepsilon_{k+1}, -k/2) \times \chi(m, 0, k/2)$$

and a submodule of

$$\chi(m, \varepsilon_{k+1}, k/2) \times \chi(m, 0, -k/2)$$

where $k \in \mathbb{N}$ and $\varepsilon_{k+1} \equiv k+1 \pmod{2}$.

Remark 4.1.3. *By Lemma 4.3.3 and Proposition 4.3.6 below, the Speh representation $\delta(2m, k)$ is the unique irreducible submodule of $\chi(m, \varepsilon_{k+1}, k/2) \times \chi(m, 0, -k/2)$ and thus also the unique irreducible quotient of $\chi(m, \varepsilon_{k+1}, -k/2) \times \chi(m, 0, k/2)$.*

4.1.2. *Vogan Classification.* By the Vogan classification [Vog86], irreducible unitary representations of G_n are BZ products of the form

$$\pi = \pi_1 \times \cdots \times \pi_k$$

where each π_i is one of the following basic unitary representations:

- (a) A one-dimensional unitary character of some G_m .
- (b) A Stein complementary series representation of some G_{2m} , twisted by a unitary character. The Stein representations are complementary series of the form

$$\sigma(2m, s) = \chi(m, 0, s) \times \chi(m, 0, -s), s \in (0, 1/2)$$

and we write $\sigma(2m, s; \varepsilon, it)$ to denote its twist by $\chi(2m, \varepsilon, it)$.

- (c) A Speh representation $\delta(2m, k)$ of G_{2m} , twisted by a unitary character. We write $\delta(2m, k; it)$ to denote its twist by $\chi(2m, 0, it)$.
- (d) A Speh complementary series representation of some G_{4m} , twisted by a unitary character. The Speh complementary series representation is

$$\Delta(4m, k, s) = \delta(2m, k; 0, s) \times \delta(2m, k; 0, -s), s \in (0, 1/2)$$

and we write $\Delta(4m, k, s; it)$ to denote its twist by $\chi(4m, 0, it)$.

It is known that the associated partition of $\delta(2m, k)$ is 2^m and (thus) the associated partition of $\Delta(4m, k, s)$ is 4^m (see [GS, Corollary 4.0.2]).

If $F = \mathbb{C}$ then only types (a) and (b) appear in the Vogan classification. Thus, every $\tau \in \widehat{GL(n, \mathbb{C})}$ is a product of (not necessary unitary) characters. For $F = \mathbb{R}$ this is not true, but the above information implies the following proposition.

Corollary 4.1.4 ([GS], Corollary 4.0.6). *Let $\tau \in \widehat{G_n}$, let λ be the associated partition of τ and $\mu = (n_1, \dots, n_k)$ be the transposed partition. Then there exist (not necessary unitary) characters χ_i, χ'_i of G_{n_i} for all $1 \leq i \leq k$, an epimorphism $\chi_1 \times \cdots \times \chi_k \twoheadrightarrow \tau^\infty$ and an embedding $\tau^\infty \hookrightarrow \chi'_1 \times \cdots \times \chi'_k$. Moreover, $\mathcal{V}(\chi_1 \times \cdots \times \chi_k) = \mathcal{V}(\chi'_1 \times \cdots \times \chi'_k) = \mathcal{V}(\tau^\infty)$.*

4.1.3. *Adduced representation.* Adduced representation was defined in [Sah89] for irreducible unitary representations. The definition is based on the fact that an irreducible unitary representation of G_n remains irreducible (as a unitary representation) after restriction to P_n . The adduced representation of $\tau \in \widehat{G_n}$, denoted $A\tau$ is defined to be the irreducible unitary representation of G_{n-d} that gives rise to $\tau|_{P_n}$ by Mackey induction. The number d is called the depth of τ . By Theorem 4.1.9 below, $\text{depth}(\tau) = \text{depth}(\tau^\infty)$.

Clearly, $A\chi = \chi|_{G_{n-1}}$ for any unitary character χ . It is also clear that A “commutes” with a twist by a unitary character. In [Sah89], it was proven that A is a multiplicative operation (an analog of Conjecture 3.1.1). In [Sah90], it was shown that $A\sigma(2m, s) = \sigma(2m - 2, s)$, which completed the computation of adduced representations for $F = \mathbb{C}$. In [SaSt90] it was shown that $A\delta(2m, k) = \delta(2m - 2, k)$. Finally, in [GS] it was shown, using the Vogan classification, that $A\Delta(4m, k, s) = \Delta(4m - 4, k, s)$ if $k \neq m$ and conjectured that the condition $k \neq m$ is not necessary. Of course, this conjecture immediately follows from Conjecture 3.1.1 and the fact that $A\delta(2m, k) = \delta(2m - 2, k)$. However, in this section we prove that $A\Delta(4m, k, s) = \Delta(4m - 4, k, s)$ for all k and m in a different way (without using the Vogan classification). This completes the computation of adduced representations for all irreducible unitary representations of G_n .

We also prove that $(A\tau)^\infty \cong B(\tau^\infty) \cong D(\tau^\infty) \cong E(\tau^\infty)$.

The following lemma follows from the definition of adduced representation and Frobenius reciprocity. We refer the reader to Notation 3.0.1 for the definitions of the groups S_n^d and U_n^d .

Lemma 4.1.5 ([GS], Proposition 3.1.2). *Let $\tau \in \widehat{G_n}$, and let $d := \text{depth}(\tau)$. Extend the action of G_{n-d} on $(A\tau)^\infty$ to an action of S_n^d by letting U_n^d act by the character ψ^d .*

Then there exists an S_n^d -equivariant map from τ^∞ to $(A\tau)^\infty \otimes |\det|^{(d-1)/2}$ with dense image.

Corollary 4.1.6. *Let $\tau \in \widehat{G_n}$, and let $d := \text{depth}(\tau)$. Then there is a natural epimorphism $B_\infty^d(\tau^\infty) \twoheadrightarrow (A\tau)^\infty$.*

Proof. Twisting the map from the previous lemma by $|\det|^{-(d-1)/2}$ we obtain a map $\tau^\infty \otimes |\det|^{-(d-1)/2} \rightarrow (A\tau)^\infty$. Since it is S_n^d -equivariant, it factors through $B_\infty^d(\tau^\infty)$. Thus we have a map $B_\infty^d(\tau^\infty) \rightarrow (A\tau)^\infty$ with dense image. By the Casselman-Wallach theorem (see Corollary 2.2.5) the image is closed and hence this is an epimorphism. \square

4.1.4. Degenerate Whittaker models. For a composition $\lambda = (n_1, \dots, n_k)$ of n , let J_λ denote the corresponding upper triangular block matrix and let $w_0 J_\lambda w_0^{-1}$ be its conjugation by the longest Weyl group element w_0 . Let ψ_λ denote the character of \mathfrak{n} given by $\psi_\lambda(X) := \text{Tr}(X w_0 J_\lambda w_0^{-1})$. By abuse of notation, we denote the corresponding character of N also by ψ_λ .

Definition 4.1.7. Let $\pi \in \mathcal{M}_\infty(G_n)$ and λ be a composition of n .

- Denote $Wh_\lambda^*(\pi) := \text{Hom}_N(\pi, \psi_\lambda)$ and for $\tau \in \widehat{G}_n$ denote $Wh_\lambda^*(\tau) := Wh_\lambda^*(\tau^\infty)$.
- Denote $E^\lambda(\pi) := E^{n_k}(\cdots(E^{n_1}(\pi)|_{\mathfrak{p}_{n-n_1}}) \cdots)$ and $B^\lambda(\pi) := B^{n_k}(\cdots(B^{n_1}(\pi)|_{\mathfrak{p}_{n-n_1}}) \cdots)$.

The following lemma is obvious.

Lemma 4.1.8.

- (1) $Wh_\lambda^*(\pi) \cong (B^\lambda(\pi))^*$
- (2) We have a natural epimorphism $E^\lambda(\pi) \rightarrow B^\lambda(\pi)$.

We will use the main result of [GS].

Theorem 4.1.9 ([GS], Theorem A). *Let $\tau \in \widehat{G}_n$ and let $\lambda = (n_1, \dots, n_k)$ be the associated partition of τ . Then*

- (1) $Wh_\lambda^*(\tau) \neq 0$.
- (2) $\text{depth}(\tau) = n_1$ and the associated partition of $A\tau$ is (n_2, \dots, n_k) .

We will show that $\dim Wh_\lambda^*(\tau) = 1$ (see Theorem 4.2.3).

4.2. Applications. We will use the following immediate corollary of Theorem 3.0.13.

Corollary 4.2.1. *Let $\alpha = (n_1, \dots, n_k)$ be a composition of n , λ be the partition obtained by reordering of α and μ be the transposed partition of λ . Let χ_i be characters of G_{n_i} for $1 \leq i \leq k$. Then $\dim E^\mu(\chi_1 \times \cdots \times \chi_k) = 1$.*

By exactness of E (Theorem 3.0.8) we obtain

Corollary 4.2.2. *In the notations of the previous corollary, $\chi_1 \times \cdots \times \chi_k$ has a unique irreducible subquotient π such that $\dim E^\mu(\pi) = 1$. For any other irreducible subquotient ρ we have $E^\mu(\rho) = B^\mu(\rho) = 0$.*

First of all, let us prove the following strengthening of Theorem 4.1.9.

Theorem 4.2.3. *Let $\tau \in \widehat{G}_n$ and let $\lambda = (n_1, \dots, n_k)$ be the associated partition of τ . Then $\dim Wh_\lambda^*(\tau) = 1$.*

Proof. Let $\mu = (m_1, \dots, m_l)$ be the transposed partition to λ . By Corollary 4.1.4, there exist characters χ_i of G_{m_i} for all $1 \leq i \leq l$ and an epimorphism $I_\mu := \chi_1 \times \cdots \times \chi_l \rightarrow \tau^\infty$. By Corollary 4.2.1, $\dim E^\lambda(I_\mu) = 1$ and hence

$$\dim Wh_\lambda^*(\tau) \leq \dim B^\lambda(\tau^\infty) \leq \dim B^\lambda(I_\mu) \leq \dim E^\lambda(I_\mu) = 1.$$

Since by Theorem 4.1.9 $Wh_\lambda^*(\tau) \neq 0$, we have $\dim Wh_\lambda^*(\tau) = 1$. \square

Let us now compute the adduced representations of the Speh complementary series representations.

Theorem 4.2.4. $A(\Delta(4m, k, s)) \cong \Delta(4m - 4, k, s)$.

Proof. Since the associated partition of $\Delta(4m, k, s)$ is 4^m , Theorem 4.1.9 implies that $\text{depth}(\Delta(4m, k, s)) = 4$ and the associated partition of $A(\Delta(4m, k, s))$ is $(4)^{m-1}$ which is the same as that of $\Delta(4m - 4, k, s)$. Let

$$I_{m,k,s} := \chi(m, \varepsilon_{k+1}, -k/2 + s) \times \chi(m, 0, k/2 + s) \times \chi(m, \varepsilon_{k+1}, -k/2 - s) \times \chi(m, 0, k/2 - s).$$

From §§4.1.2 we see that $\Delta(4m, k, s)$ is a quotient of $I_{m,k,s}$. Thus, $E_\infty^4(\Delta(4m, k, s))$ is a quotient of $E_\infty^4(I_{m,k,s})$. By Theorem 3.0.13, $E_\infty^4(I_{m,k,s}) = I_{m-1,k,s}$ and by Corollary 4.1.6 $A(\Delta(4m, k, s))$ is a quotient of $E_\infty^4(\Delta(4m, k, s)) \otimes |\det|^{1/2}$. Altogether, we get that $A\Delta(4m, k, s)$ is a quotient of $I_{m-1,k,s}$. But so is $\Delta(4m-4, k, s)$.

From Theorem 4.1.9 we have $Wh_{4m-1}^*(A(\Delta(4m, k, s))) \neq 0$ and $Wh_{4m-1}^*(\Delta(4m-4, k, s)) \neq 0$. Thus, Corollary 4.2.2 implies $A(\Delta(4m, k, s)) \cong \Delta(4m-4, k, s)$. \square

Remark 4.2.5. *Using a similar argument, one can give an alternative proof of $A\delta(2m, k) \cong \delta(2m-2, k)$, using only the fact that $\delta(2m, k)$ is an irreducible quotient of $\chi(m, \varepsilon_{k+1}, -k/2) \times \chi(m, 0, k/2)$ with associated partition 2^m .*

4.3. The isomorphisms between depth derivatives and adduced representation. In this subsection we prove that for irreducible unitary representations, all the notions of highest derivative agree and extend the notion of adduced representation.

Theorem 4.3.1. *For $\tau \in \widehat{G}_n$ of depth d , the canonical maps*

$$E_\infty^d(\tau^\infty) \rightarrow D_\infty^d(\tau^\infty) \rightarrow B_\infty^d(\tau^\infty) \rightarrow (A\tau)^\infty$$

are isomorphisms.

For case $F = \mathbb{C}$ it follows from the Vogan classification (§§4.1.2), Theorem 3.0.13 and Corollary 4.1.6. Thus from now on till the end of the section, we let $F = \mathbb{R}$. It is enough to prove that the resulting map $E_\infty^d(\tau^\infty) \rightarrow (A\tau)^\infty$ has zero kernel.

Definition 4.3.2. *To every $\tau \in \widehat{G}_n$ of depth d we attach a monomial representation $I^\geq(\tau)$. First, we attach to τ d characters using the Vogan classification in the following way. To every character we attach itself, to the Speh representation $\delta(m, k)$ we attach $\chi(m, \varepsilon_{k+1}, k/2)$ and $\chi(m, 0, -k/2)$ and to the Speh complementary series $\Delta(m, k, s)$ we attach $\chi(m, \varepsilon_{k+1}, k/2+s)$, $\chi(m, 0, -k/2+s)$, $\chi(m, \varepsilon_{k+1}, k/2-s)$ and $\chi(m, 0, -k/2-s)$. Then we order all the characters we have in non-ascending order of the real part of the complex parameter, and let $I^\geq(\tau)$ be their BZ product in this order.*

Note that by Proposition 4.1.2 and Corollary 4.1.4, τ^∞ is a subquotient of $I^\geq(\tau)$ and $\mathcal{V}(\tau) = \mathcal{V}(I^\geq(\tau))$.

Lemma 4.3.3. *Every $\tau^\infty \in \widehat{G}_n$ occurs with multiplicity one in $I^\geq(\tau)$ and it is the only irreducible subquotient of $I^\geq(\tau)$ with maximal annihilator variety.*

Proof. Let δ denote a Speh representation. By Proposition 4.1.2, $\delta \subset I^\geq(\delta)$ and by [BSS90, Proposition VI.7], $\mathcal{V}(I^\geq(\delta)/\delta) \subsetneq \mathcal{V}(I^\geq(\delta))$. Now let $\tau \in \widehat{G}_n$. Then $\tau = \prod \chi_i \times \prod \delta_j$, where χ_i are characters, and δ_j are Speh representations, possibly twisted by characters. Then, by exactness of tensor product (see Proposition 9.4.4 below), $(\boxtimes_i \chi_i) \boxtimes (\boxtimes_j I^\geq(\delta_j)) \subset (\boxtimes_i \chi_i) \boxtimes (\boxtimes_j \delta_j)$ and the annihilator variety of the quotient is strictly smaller than the annihilator variety of $(\boxtimes_i \chi_i) \boxtimes (\boxtimes_j \delta_j)$. The lemma now follows from the exactness of induction, Theorem 2.4.2 and the fact that $I^\geq(\tau)$ and $\prod \chi_i \times \prod I^\geq(\delta_j)$ have the same Jordan-Holder components. \square

Remark 4.3.4. *One can show that any monomial representation has a unique subquotient with maximal annihilator variety, occurring with multiplicity one. By [GS2], this is equivalent to the statement that it has a unique subquotient with minimally degenerate Whittaker functional, which follows from Corollary 4.2.2.*

Another possible way to prove that is by using a more refined invariant than \mathcal{V} , called the associated cycle. It is a nonnegative integer combination of nilpotent coadjoint orbits, and it is additive on the Grothendieck group. The fact that the moment map from $T^(G/P) \rightarrow \mathfrak{g}^*$ is generically one to one in G_n probably implies that the associated cycle of a monomial representation μ is 1 times the corresponding nilpotent orbit. From these two facts it follows that μ has a unique large constituent, having multiplicity one.*

Lemma 4.3.5. *For $\tau \in \widehat{G}_n$ of depth d , $I^\geq(A\tau) = E^d(I^\geq(\tau))$.*

This lemma follows from Theorem 3.0.13, §§§4.1.3 and Theorem 4.2.4.

We will also use the following proposition that we will prove in §§§4.3.1.

Proposition 4.3.6. *Let χ_1, \dots, χ_k be characters of G_{n_1}, \dots, G_{n_k} respectively and let $d\chi_i$ denote the corresponding characters of \mathfrak{g}_{n_i} . Suppose that $\text{Re } d\chi_1 \geq \dots \geq \text{Re } d\chi_k$. Then any subrepresentation $\pi \subset \chi_1 \times \dots \times \chi_k$ satisfies $\mathcal{V}(\pi) = \mathcal{V}(\chi_1 \times \dots \times \chi_k)$.*

Proof of Theorem 4.3.1. Let ρ be the kernel of the canonical map $E^d(\tau^\infty) \rightarrow A\tau^\infty$ (given by Corollary 4.1.6). We have to show that ρ is zero.

By Lemma 4.3.3 and Proposition 4.3.6 we have an embedding $\tau^\infty \subset I^\geq(\tau)$. By Theorem 3.0.8 and Lemma 4.3.5 we have $E^d(\tau^\infty) \subset E^d(I^\geq(\tau)) = I^\geq(A\tau)$. Thus, we get that ρ is a subrepresentation of $I^\geq(A\tau)$ and thus, by Proposition 4.3.6, either $\rho = 0$ or $\mathcal{V}(\rho) = \mathcal{V}(I^\geq(A\tau)) = \mathcal{V}(A\tau)$. The second option contradicts Lemma 4.3.3 and thus $\rho = 0$. \square

4.3.1. *Proof of Proposition 4.3.6.* We will use the following lemmas.

Lemma 4.3.7. *Let \mathfrak{q} be a standard parabolic subgroup of \mathfrak{g}_n with nilradical \mathfrak{u} and Levi subgroup \mathfrak{l} . Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a character of \mathfrak{l} . Let $M_\lambda := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} (\lambda - \rho(\mathfrak{u}))$ be the corresponding (normalized) generalized Verma module. Suppose that $\lambda_1 \geq \dots \geq \lambda_k$. Then M_λ is irreducible.*

This follows e.g. from [Tra86, Lemma 3.5] or [Jan77, Satz 4 and Corollar 4].

Lemma 4.3.8. *[see e.g. [GS], Lemma 2.3.9] Let G and Q be Lie groups, σ be a smooth representation of Q and $(\pi, W) = \text{Ind}_Q^G(\sigma)$ be the (normalized) induction of σ . Suppose that G/Q is compact and connected and $(\sigma \otimes \Delta_Q^{1/2} \otimes \Delta_G^{-1/2})^\omega$ is non-degenerately \mathfrak{q} -invariantly paired with a \mathfrak{q} -module X . Then π^ω is non-degenerately \mathfrak{g} -invariantly paired with a quotient of $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} X$, where ω denotes the space of analytic vectors.*

Proof of Proposition 4.3.6. Let $I = \chi_1 \times \dots \times \chi_k$, $Q := P_{(n_1, \dots, n_k)}$ and $\lambda := (\chi_1, \dots, \chi_k)$. Let $M_\lambda := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} (\lambda - \rho(\mathfrak{u}))$ be the corresponding (normalized) generalized Verma module. By Lemma 4.3.7 and the assumptions of the proposition, M_λ is irreducible. Thus, by Lemma 4.3.8, M_λ is non-degenerately paired with I^ω and hence also with π^ω for any submodule $\pi \subset I$.

Note that annihilators of two non-degenerately paired representations are equal and that $\text{Ann}(\pi) = \text{Ann}(\pi^\omega) = \text{Ann}(\pi^{HC})$, since π^{HC} includes π^ω and is dense in π . Thus, the annihilators of π , M_λ and I are equal and thus $\mathcal{V}(\pi) = \mathcal{V}(M_\lambda) = \mathcal{V}(I)$. \square

5. PROOF OF ADMISSIBILITY (THEOREM 3.0.5)

5.1. Structure of the proof. First of all, by Theorem 2.2.2, a (\mathfrak{g}, K) -module π is admissible if and only if it is finitely generated over \mathfrak{n} . Thus, we know that $E^d(\pi)$ is finitely generated over \mathfrak{n}_{n-d+1} and we need to show that it is in fact finitely generated over \mathfrak{n}_{n-d} .

Let us consider an analogous situation in commutative algebra. Let W be a finite-dimensional complex vector space, and $W = U \oplus V \oplus W'$ be a direct sum decomposition. Let $\xi \in U^*$. Note that it defines a homomorphism of algebras $\Xi : \text{Sym}(W) \rightarrow \text{Sym}(V \oplus W')$. The corresponding map $(V \oplus W')^* = \text{Spec}(\text{Sym}(V \oplus W')) \rightarrow \text{Spec}(\text{Sym}(W)) = W^*$ is given just by adding ξ . Let $I := \text{Ker } \Xi$. Let M be a finitely-generated module over $\text{Sym}(W)$ and $L := M/I$. Suppose that we know the support of M and want to find out when L is finitely generated over $\text{Sym}(W')$. A sufficient condition will be $\text{Supp } M \subset \xi + (W')^* + 0$. Indeed, this condition is in turn equivalent to $\text{pr}_V(\text{Supp } L) = 0$, which implies that V acts on L nilpotently and thus is not needed for finite generation.

Our proof follows the general lines of this argument. However, over non-commutative Lie algebras we have several problems. The first is that we do not have a straightforward notion of support. One substitute is the annihilator variety. As we recall in §§2.3, it is defined by considering the annihilator ideal of the module in $\mathcal{U}(\mathfrak{g})$, then passing to the graded ideal in $\text{Sym}(\mathfrak{g})$ and considering its zero set. In our proof we also use a second notion of support, called associated variety, that we recall in §§5.2. It is a finer invariant, defined by first passing to associated graded module and then considering the support in $\text{Spec } \mathfrak{g}^* = \text{Sym}(\mathfrak{g})$. In order to define this invariant, one has to fix a filtration on the module M . If M is

finitely generated, it has so-called “good” filtrations (see §§2.3) and the associated variety $AV(M)$ does not depend on the choice of a good filtration.

For finitely-generated modules over commutative Lie algebras, $\mathcal{V}(M) = AV(M)$. However, they might be different from $\text{Supp}(M)$, because AV and \mathcal{V} are always conical sets. For example, let $\mathfrak{g} = \mathbb{C}$, $U(\mathfrak{g}) = \text{Sym}(\mathfrak{g}) = \mathbb{C}[x]$ and $M := \mathbb{C}[x]/(x-1)$. Then $\text{Supp}(M) = \{1\}$ and $AV(M) = \mathcal{V}(M) = 0$. Moreover, over non-commutative algebras the associated variety of a quotient is not determined by the associated variety of M .

Thus, we will have to use the fact that our module comes from \mathfrak{g}_n , and thus its annihilator ideal is invariant under the torus action. Thus, it is a graded ideal with respect to the grading defined by any torus element. In order to connect the associated variety with respect to actions of \mathfrak{g}_n and \mathfrak{s}_n we will use a proposition saying that any good \mathfrak{g}_n -filtration on π is also a good \mathfrak{n}_n -filtration (see Proposition 5.2.7).

Another difficulty we have in the non-commutative case is that the quotient of $\pi \otimes \psi^d$ by \mathfrak{u}^d is not a module over $\mathfrak{s}_d/\mathfrak{u}^d = \mathfrak{g}_{n-d+1}$, but only over \mathfrak{p}_{n-d+1} . In order to cope with this difficulty we use a second torus element, and thus consider a bigrading on $U(\mathfrak{g}_n)$.

We introduce the required notations and terminology on bigradings in §§5.3 and on the module of coinvariants in §§5.4. In §§5.5 we prove the algebraic key lemma that connects the associated variety of $E^d(\pi)$ to the annihilator variety of π . This lemma is a generalization of [Mat87, Theorem 1]. Finally, in §§5.6 we deduce that $\text{pr}_{V_{n-d+1}^*}(AV(E^d(\pi))) = \{0\}$ and finish the proof of the theorem.

5.2. Filtrations and associated varieties. In this subsection we define the associated variety of a (finitely generated) module over a Lie algebra. For that we will need the notion of filtration. Throughout the subsection, let A denote a not necessary commutative algebra with 1 over \mathbb{C} .

Definition 5.2.1. *A filtration on A is a presentation of A as an ascending union of vector spaces $A = \bigcup_{i \geq 0} F^i A$ such that $F^i A F^j A \subset F^{i+j} A$. A filtration on an A -module M is a presentation of M as an ascending union of vector spaces $M = \bigcup_{i \geq 0} F^i M$ such that $F^i A F^j M \subset F^{i+j} M$.*

If A is a filtered algebra we define the associated graded algebra $\text{Gr}(A) := \bigoplus_{i \geq 0} F^{i+1} A / F^i A$ with the obvious algebra structure. If M is a filtered module we define the associated graded module $\text{Gr}(M)$ over $\text{Gr}(A)$ by $\text{Gr}(M) := \bigoplus_{i \geq 0} F^{i+1} M / F^i M$.

Definition 5.2.2. *Let A be a filtered algebra and M be an A -module. A filtration $F^i M$ is called good if*

- (i) *Every $F^i M$ is a finitely generated module over $F^0 A$ and*
- (ii) *There exists n such that for every $i > n$, $F^{i+1} M = F^1 A F^i M$.*

A filtration on A is called good if it is a good filtration of A as a module over itself.

Example 5.2.3. *Let \mathfrak{g} be a (finite dimensional) Lie algebra and $U(\mathfrak{g})$ the universal enveloping algebra. Define a good filtration U^i on $U(\mathfrak{g})$ by the order of the tensor. Then $\text{Gr}(U(\mathfrak{g})) = \text{Sym}(\mathfrak{g})$, the symmetric algebra.*

From now on we fix a good filtration on A . Let M be an A -module.

Example 5.2.4. *Suppose that M is finitely generated. Let $\{m_1, \dots, m_k\}$ be a set of generators. We define a good filtration on M by $F^i M = \{\sum_{j=1}^k a_j m_j \text{ s.t. } a_j \in F^i A\}$.*

The following lemma is evident.

Lemma 5.2.5.

- (i) *There exists a good filtration on M if and only if M is finitely generated over A .*
- (ii) *A filtration $F^i M$ is good if and only if $\text{Gr}_F(M)$ is finitely generated over $\text{Gr}(A)$.*

Corollary 5.2.6. *Suppose that $\text{Gr}(A)$ is Noetherian. Suppose that M is finitely generated and let $F^i M$ be a good filtration on M . Then*

- (i) *For any submodule $L \subset M$, the induced filtration $F^i L := F^i M \cap L$ is good.*
- (ii) *A is Noetherian.*

In particular, $U(\mathfrak{g})$ is Noetherian.

An important tool in the proof of Theorem 3.0.5 will be the following proposition.

Proposition 5.2.7. *Let $\pi \in \mathcal{M}_{HC}(G)$ be a Harish-Chandra module and let F^i be a good \mathfrak{g} -filtration on it. Then F^i is good as an \mathfrak{n} -filtration, i.e. $F^{i+1} = \mathfrak{n}F^i$ for i big enough.*

This proposition is due to Gabber and is based on a proposition by Casselman and Osborne. For completeness we included its proof in Appendix B.

Definition 5.2.8. *For any filtration F on M we can associate to M a subvariety of $\text{Spec } \text{Gr}(A)$ by $\text{AV}_F(M) := \text{Supp}(\text{Gr}(M)) \subset \text{Spec } \text{Gr}(A)$.*

If M is finitely generated we choose a good filtration F on M and define the associated variety of M to be $\text{AV}(M) := \text{AV}_F(M)$. This variety does not depend on the choice of the good filtration.

Remark. *If A is commutative then the associated variety equals to the annihilator variety. Otherwise, the associated variety can be smaller.*

5.3. Bigraded Lie algebra. Let \mathfrak{g} be a Lie algebra and let $X, Y \in \mathfrak{g}$ be commuting ad-semisimple elements with integer eigenvalues. Define

$$\mathfrak{g}_{ij} := \{Z \in \mathfrak{g} : [X, Z] = iZ, [Y, Z] = jZ\}$$

Thus we have the direct sum decomposition

$$(1) \quad \mathfrak{g} = \bigoplus_i \mathfrak{g}_i \text{ where } \mathfrak{g}_i = \bigoplus_j \mathfrak{g}_{ij}$$

We now choose an ordered basis \mathcal{B} of \mathfrak{g} as follows. Pick ordered bases \mathcal{B}_{ij} for each \mathfrak{g}_{ij} , order the pairs (i, j) lexicographically so that

$$(2) \quad (i, j) \succ (k, l) \text{ if } i > k \text{ or if } i = k \text{ and } j > l$$

and let \mathcal{B} be the descending union of \mathcal{B}_{ij} . Thus \mathcal{B} is ordered so that \mathcal{B}_{ij} precedes \mathcal{B}_{kl} if $(i, j) \succ (k, l)$. By the Poincare-Birkhoff-Witt theorem the corresponding ordered (PBW) monomials form a basis for the enveloping algebra $\mathcal{U}(\mathfrak{g})$.

Definition 5.3.1. *If $u \in \mathcal{U}(\mathfrak{g})$ and a PBW monomial T has a nonzero coefficient in the expansion of u , we say u contains T .*

Note that adX, adY act on $\mathcal{U}(\mathfrak{g})$ with integer eigenvalues as well and we define

$$(3) \quad \mathcal{U}_{ij}(\mathfrak{g}) = \{u \in \mathcal{U}(\mathfrak{g}) : [X, u] = iu, [Y, u] = ju\}$$

By construction each PBW monomial belongs to some $\mathcal{U}_{ij}(\mathfrak{g})$, and thus the following result is obvious.

Lemma 5.3.2. *If $u \in \mathcal{U}_{ij}(\mathfrak{g})$ and u contains T , then $T \in \mathcal{U}_{ij}(\mathfrak{g})$.*

5.4. Conivariants module. For $s \geq 0$ define $\mathcal{N}_s = \bigoplus_{i \geq s} \mathfrak{g}_i$; then \mathcal{N}_0 is a Lie subalgebra and \mathcal{N}_1 is a nilpotent ideal of \mathcal{N}_0 . Let $\xi \in \mathcal{N}_1^*$ be such that $\xi|_{\mathcal{N}_2} = 0$ then ξ defines a Lie algebra character of \mathcal{N}_1 , and we have

$$\mathcal{N}_0^\xi = \mathfrak{g}_0^\xi \oplus \mathcal{N}_1$$

where \mathcal{N}_0^ξ and \mathfrak{g}_0^ξ denote the stabilizers of ξ in the respective Lie algebras.

Consider the linear map $\Xi : \mathcal{N}_0^\xi \rightarrow \mathcal{U}(\mathfrak{g}_0^\xi)$ given by

$$(4) \quad \Xi(Z) = \begin{cases} Z & \text{if } Z \in \mathfrak{g}_0^\xi \\ \xi(Z) & \text{if } Z \in \mathcal{N}_1 \end{cases}$$

It is easy to check that Ξ is a Lie algebra map, i.e. it intertwines the Lie bracket with the commutator, and hence by universality it extends to an algebra map from $\mathcal{U}(\mathcal{N}_0^\xi)$ to $\mathcal{U}(\mathfrak{g}_0^\xi)$ that we continue to denote by Ξ .

Suppose M is an \mathcal{N}_0^ξ -module, we define the ξ -coinvariant module to be

$$L = M/M' \text{ where } M' = \text{span} \{Zv - \xi(Z)v \mid Z \in \mathcal{N}_1, v \in M\}$$

Then L is a \mathfrak{g}_0^ξ -module and the projection map $\varpi : M \rightarrow L$ satisfies

$$(5) \quad \varpi(uv) = \Xi(u)\varpi(v) \text{ for } u \in \mathcal{U}(\mathcal{N}_0^\xi), v \in M.$$

5.5. The Algebraic Key Lemma. In this subsection we assume the following.

Condition 5.5.1.

- (1) \mathfrak{g} is a Lie algebra with elements X, Y and bigrading \mathfrak{g}_{ij} as in §§5.3.
- (2) $\mathfrak{g}_{ij} = \{0\}$ if $j \notin \{-1, 0, 1\}$ and also that $\mathfrak{g}_{1,-1} = \{0\}$.
- (3) ξ is a character of \mathcal{N}_1 as in §§5.4 and $\xi|_{\mathfrak{g}_{ij}} = 0$ unless $i = 1, j = 0$.

Lemma 5.5.2. Suppose \mathfrak{g} and ξ satisfy (1)-(3) of Condition 5.5.1 then we have $\mathfrak{g}_{0,1} \subset \mathfrak{g}_0^\xi$.

Proof. We need to show that $\xi([A, B]) = 0$ for all $A \in \mathfrak{g}_{0,1}, B \in \mathcal{N}_1$.

To prove this we may assume that $B \in \mathfrak{g}_{ij}$ for some i, j . Then we have $[A, B] \in \mathfrak{g}_{i,j+1}$ and so by Condition 5.5.1 (3) we have $\xi([A, B]) = 0$ unless $i = 1$ and $j = -1$. This forces $B \in \mathfrak{g}_{1,-1}$ and hence $B = 0$ by Condition 5.5.1 (2) and so $\xi([A, B]) = 0$ in this case as well. \square

In view of the previous lemma we have a well-defined restriction map

$$\text{Res} : (\mathfrak{g}_0^\xi)^* \rightarrow (\mathfrak{g}_{0,1})^*$$

Let M be a \mathfrak{g} -module and fix a (not necessary good) filtration $F^i M$ on M . We now define the ξ -coinvariants module L of M as in §§5.4 and let $F^i L$ be the induced filtration on it.

Let

$$AV_F(L) \subset (\mathfrak{g}_0^\xi)^* \text{ and } \mathcal{V}(M) \subset \mathfrak{g}^*$$

denote the respective F -associated variety of L and annihilator variety of M as in §§5.2 and §§§2.3.

We are now ready to formulate the algebraic key lemma.

Lemma 5.5.3 (The algebraic key lemma). Suppose $\phi \in \text{Res}[AV_F(L)] \subset (\mathfrak{g}_{0,1})^*$ and regard $\phi + \xi$ as an element of \mathfrak{g}^* via

$$(\phi + \xi)|_{\mathfrak{g}_{0,1}} = \phi, (\phi + \xi)|_{\mathfrak{g}_{1,0}} = \xi \text{ and } (\phi + \xi)|_{\mathfrak{g}_{ij}} = 0 \text{ for all other pairs } (i, j)$$

Then we have

$$\phi + \xi \in \mathcal{V}(M)$$

The proof involves in a crucial way the PBW basis discussed in §§5.3 above. Note that by Condition 5.5.1 (2), the sequence of pairs (i, j) ordered as in (2) looks as follows:

$$(6) \quad \boxed{\dots, (1, 1), (1, 0), (0, 1), (0, 0), (0, -1), (-1, 1), \dots}$$

where we have grouped the possible pairs (i, j) into 4 groups for ease of reference below. Note that we do not mean to imply that $\mathcal{B}_{ij} \neq \emptyset$ for the indicated pairs in (6), but rather that $\mathcal{B}_{ij} = \emptyset$ for the missing pairs e.g. $(1, -1), (0, 2)$ etc.

Proof of Lemma 5.5.3. Let $\sigma^n : \mathcal{U}^n(\mathfrak{g}) \rightarrow \text{Sym}^n(\mathfrak{g})$ denote the n -th symbol map. We need to show that for all n , and for all $P \in \text{Ann}(M) \cap \mathcal{U}^n(\mathfrak{g})$ we have

$$(7) \quad \langle \sigma^n(P), \phi + \xi \rangle = 0$$

Since $\text{Ann}(M)$ and $\mathcal{U}^n(\mathfrak{g})$ are stable under the adjoint action ad , $\text{Ann}(M) \cap \mathcal{U}^n(\mathfrak{g})$ is a direct sum of $ad(X)$ -eigenspaces. Since X and Y commute, each $ad(X)$ -eigenspace in $\text{Ann}(M) \cap \mathcal{U}^n(\mathfrak{g})$ is a direct sum of $ad(Y)$ -eigenspaces. Thus we may further assume

$$P \in \text{Ann}(M) \cap \mathcal{U}^n(\mathfrak{g}) \cap \mathcal{U}_{kl}(\mathfrak{g})$$

for some integers k, l , where $\mathcal{U}_{kl}(\mathfrak{g})$ is defined as in (3).

Consider the PBW monomials contained in P in the sense of Definition 5.3.1. We say such a monomial is “relevant” if it is a product of precisely n factors from group 2 in the sequence (6) (i.e. $\{(1, 0), (0, 1)\}$) and “irrelevant” otherwise. Thus we get a decomposition

$$P = R + I$$

where R and I are combinations of relevant and irrelevant monomials respectively.

We note that $R \in \mathcal{U}(\mathcal{N}_0^\xi)$ and we claim that the following properties hold

$$(8) \quad \langle \sigma^n(P), \phi + \xi \rangle = \langle \sigma^n(R), \phi + \xi \rangle$$

$$(9) \quad \Xi(R) \in \mathcal{U}^{n-k}(\mathfrak{g}_{0,1})$$

$$(10) \quad \sigma^{n-k}(\Xi(R)) \in \text{Ann}(Gr_F(L))$$

$$(11) \quad \langle \sigma^n(R), \phi + \xi \rangle = \langle \sigma^{n-k}(\Xi(R)), \phi \rangle$$

Granted these claims for the moment, we can prove the Lemma as follows. Since $\phi \in \text{Res}[AV_F(L)]$ we deduce from (9) and (10) that $\langle \sigma^{n-k}(\Xi(R)), \phi \rangle = 0$. Now by (8) and (11) we get (7) as desired.

We now turn to the proof of claims (8 – 11). For (8) its suffices to show that

$$(12) \quad \langle \sigma^n(T), \phi + \xi \rangle = 0$$

for every *irrelevant* monomial T contained in P . Indeed if T has fewer then n factors then $\sigma^n(T) = 0$, otherwise T must have a factor not from group 2 and then (12) holds since $\phi + \xi$ vanishes on such factors by definition.

If $R = 0$ then certainly (9 – 11) hold. Therefore we may assume that P contains at least one relevant monomial S . By definition every such S is of the form

$$S = A_1 \cdots A_p B_1 \cdots B_{n-p} \text{ with } A_i \in \mathcal{B}_{1,0} \text{ and } B_j \in \mathcal{B}_{0,1}$$

By Lemma 5.3.2 we have $S \in \mathcal{U}_{kl}(\mathfrak{g})$ which forces

$$(13) \quad k, l \geq 0 \text{ and } n = k + l.$$

and that S is necessarily of the form

$$(14) \quad S = A_1 \cdots A_k B_1 \cdots B_{n-k} \text{ with } A_i \in \mathcal{B}_{1,0} \text{ and } B_j \in \mathcal{B}_{0,1}$$

Now by (4) we get

$$(15) \quad \Xi(S) = \Xi(A_1 \cdots A_k B_1 \cdots B_{n-k}) = \xi(A_1) \cdots \xi(A_k) B_1 \cdots B_{n-k} \in \mathcal{U}^{n-k}(\mathfrak{g}_{0,1})$$

Since R is a combination of relevant monomials (9) follows.

To prove (10) we need to show that

$$\Xi(R) L^i \subset L^{i+n-k-1}$$

By formula (5) we have

$$\Xi(R) L^i = \Xi(R) \varpi(M^i) = \varpi(RM^i) = \varpi((P - I)M^i).$$

Since $P \in \text{Ann}(M)$ we have $PM^i = 0$ and so it suffices to show that

$$(16) \quad \varpi(TM^i) \subset L^{i+n-k-1}$$

for every *irrelevant* monomial T contained in P . For this we consider several cases.

First suppose T has a group 1 factor, then we can write $T = ZT'$ where Z is a group 1 basis vector and T' is a smaller PBW monomial. In this case we have $\xi(Z) = 0$ and hence we get

$$\varpi(TM^i) = \varpi(ZT'M^i) = \xi(Z) \varpi(T'M^i) = 0$$

which certainly implies (16).

Thus we may suppose T has no group 1 factors. It follows then that the only possible factors of T with positive *ad X* weight are those from $\mathcal{B}_{1,0}$. Now suppose that T has a group 4 factor. Since such a factor has negative *ad X* weight and so since T has *ad X* weight k , T must have at least $k + 1$ factors from $\mathcal{B}_{1,0}$. Thus $T = A_1 \cdots A_{k+1} T'$ where $A_i \in \mathcal{B}_{1,0}$ and $T' \in \mathcal{U}^{n-k-1}(\mathfrak{g})$. Thus we get

$$\varpi(TM^i) = \xi(A_1) \cdots \xi(A_{k+1}) \varpi(T'M^i) \subset L^{i+n-k-1}$$

Therefore we may assume that T has only group 2 and group 3 factors. Since $T \in \mathcal{U}_{kl}(\mathfrak{g})$ it follows that T must have exactly k factors from $\mathcal{B}_{1,0}$ and *at least* l factors from $\mathcal{B}_{0,1}$. Since T has at most n factors and $k + l = n$, it follows that T has *exactly* l factors from $\mathcal{B}_{0,1}$. Hence T is relevant, contrary to assumption. This finishes the proof of (16).

Finally to prove (11) it suffices to show that

$$\langle \sigma^n(S), \phi + \xi \rangle = \langle \sigma^{n-k}(\Xi(S)), \phi \rangle$$

for every relevant monomial $S = A_1 \cdots A_k B_1 \cdots B_{n-k}$ as in (14). For this we calculate as follows

$$\langle \sigma^n(S), \phi + \xi \rangle = \langle \xi(A_1) \cdots \xi(A_k) \sigma^{n-k}(B_1 \cdots B_{n-k}), \phi \rangle = \langle \sigma^{n-k}(\Xi(S)), \phi \rangle$$

□

5.6. Proof of Theorem 3.0.5.

In the notations of Theorem 3.0.5: It is enough to show that $\tilde{E}^d(\pi)$ is admissible. Let $(\tau, L) := \tilde{E}^d(\pi) = \pi_{\mathfrak{u}^{d-1}, \psi}$, considered as a representation of $\mathfrak{p}_{n-d+1} = \mathfrak{g}_{n-d} \ltimes \mathfrak{v}_{n-d+1}$. Denote by ϖ the projection $\pi \rightarrow \tau$.

We will need the following lemma from linear algebra.

Lemma 5.6.1. *Let $u \in \mathfrak{u}_n^d$ be the matrix that has 1s on the superdiagonal of the lower block and 0s elsewhere. Then for any $v \in \mathfrak{v}_{n-d+1}$, if $(u+v)^d = 0$ then $u = 0$.*

Proof. Let $A := u + v$. Computing A^k by induction for $k \leq d$ we see that its first $n-d$ columns will be zero, the $n-d+k$ -th column of the submatrix consisting of the first $n-d$ rows will be v , the other columns of this submatrix will be zero and the square submatrix formed by the last d rows and columns will be J_d^k , where J_d is the (upper triangular) Jordan block. Thus, $A^d = 0$ if and only if $v = 0$. □

Fix a good filtration π^i on π . Note that by Proposition 5.2.7 it will also be good as an \mathfrak{n} -filtration and define $L^i := \varpi(\pi^i)$. Note that L^i is a good filtration.

Corollary 5.6.2. $pr_{\mathfrak{v}_{n-d+1}^*}(AV(L)) = \{0\}$.

Proof. Let $\mathfrak{g} := \mathfrak{g}_n$ and let $X, Y \in \mathfrak{g}$ be diagonal matrices given by

$$X = \text{diag}(0^{n-d}, 1, 2, \dots, d) \text{ and } Y = \text{diag}(0^{n-d-1}, 1^{d+1}).$$

Consider the bigrading $\mathfrak{g} = \bigoplus_{ij} \mathfrak{g}_{ij}$ defined as in §5.3. Note that $\mathcal{N}_1 = \mathfrak{u}_d$ and let $\psi = \psi^d$. Note that the conditions of Condition 5.5.1 are satisfied.

Let $\phi \in pr_{\mathfrak{v}_{n-d+1}^*}(AV(L))$. By the algebraic key Lemma 5.5.3, we have $\phi + \psi \in \mathcal{V}(\pi)$. By the definition of d , this implies $((\phi + \psi)^*)^d = 0$ and thus, by Lemma 5.6.1, $\phi = 0$. □

Proof of Theorem 3.0.5. By Corollary 5.6.2, $pr_{\mathfrak{v}_{n-d+1}^*}(AV(L)) = \{0\}$. Hence any $X \in \mathfrak{v}_{n-d+1}$ vanishes on $AV(L) \subset \mathfrak{p}_{n-d+1}^*$. By Hilbert's Nullstellensatz this implies that there exists k such that $X^k \in \text{Ann}(\text{gr}(L))$. Since \mathfrak{v}_{n-d+1} is finite dimensional, one can find one k suitable for all $X \in \mathfrak{v}_{n-d+1}$. Since L^i is an \mathfrak{n}_{n-d+1} -good filtration on L , $\text{gr}(L)$ is finitely generated over $\text{Sym}(\mathfrak{n}_{n-d+1})$. Since $\text{Sym}^{>k}(\mathfrak{v}_{n-d+1})$ acts by zero, $\text{gr}(L)$ is finitely generated even over $\text{Sym}(\mathfrak{n}_{n-d})$. Thus, L^i is an \mathfrak{n}_{n-d} -good filtration and hence L is finitely generated over \mathfrak{n}_{n-d} . Thus, by Theorem 2.2.2 it is an admissible Harish-Chandra module over G_{n-d} . □

6. PRELIMINARIES ON SCHWARTZ FUNCTIONS ON NASH MANIFOLDS

In the proofs of Theorems 3.0.8 and 3.0.13 we will use the language of Schwartz functions on Nash manifolds, as developed in [AG08].

Nash manifolds are smooth semi-algebraic manifolds. In sections 7 and 8 only algebraic manifolds are considered and thus the reader can safely replace the word “Nash” by “real algebraic”. However, in section 9 we will use Nash manifolds which are not algebraic.

Schwartz functions are functions that decay, together with all their derivatives, faster than any polynomial. On \mathbb{R}^n it is the usual notion of Schwartz function. We also need the notion of tempered functions, i.e. smooth functions that grow not faster than a polynomial, and so do all their derivatives. For precise definitions of those notions we refer the reader to [AG08]. In this section we summarize some elements of the theory of Schwartz functions. We will give more details in §§9.6.

Notation 6.0.1. *Let X be a Nash manifold. Denote by $\mathcal{S}(X)$ the Fréchet space of Schwartz functions on X . For any Nash vector bundle \mathcal{E} over X we denote by $\mathcal{S}(X, \mathcal{E})$ the space of Schwartz sections of \mathcal{E} .*

Proposition 6.0.2 ([AG08], Theorem 5.4.3). *Let $U \subset X$ be a (semi-algebraic) open subset, then*

$$\mathcal{S}(U, \mathcal{E}) \cong \{\phi \in \mathcal{S}(X, \mathcal{E}) \mid \phi \text{ is 0 on } X \setminus U \text{ with all derivatives}\}.$$

Notation 6.0.3. *Let Z be a locally closed semi-algebraic subset of a Nash manifold X . Let \mathcal{E} be a Nash bundle over X . Denote*

$$\mathcal{S}_X(Z, \mathcal{E}) := \mathcal{S}(X - (\overline{Z} - Z), \mathcal{E}) / \mathcal{S}(X - \overline{Z}, \mathcal{E}).$$

Here we identify $\mathcal{S}(X - \overline{Z}, \mathcal{E})$ with a closed subspace of $\mathcal{S}(X - (\overline{Z} - Z), \mathcal{E})$ using the description of Schwartz functions on an open set (Proposition 6.0.2).

Corollary 6.0.4. *Let $X := \bigcup_{i=1}^k X_i$ be a Nash stratification of a Nash manifold X , i.e. $\bigcup_{i=j}^k X_i$ is an open Nash subset of X for any j . Let \mathcal{E} be a Nash bundle over X .*

Then $\mathcal{S}(X, \mathcal{E})$ has a natural filtration of length k such that $Gr^i(\mathcal{S}(X, \mathcal{E})) = \mathcal{S}_X(X_i, \mathcal{E})$.

Moreover, if Y is a Nash manifold and $X \subset Y$ is a (locally closed) Nash submanifold then $\mathcal{S}_Y(X, \mathcal{E})$ has a natural filtration of length k such that $Gr^i(\mathcal{S}_Y(X, \mathcal{E})) = \mathcal{S}_Y(X_i, \mathcal{E})$.

Notation 6.0.5. *Let X be a Nash manifold.*

(i) *We denote by D_X the Nash bundle of densities on X . It is the natural bundle whose smooth sections are smooth measures. For the precise definition see e.g. [AG10].*

(ii) *Let $\phi : X \rightarrow Y$ be a map of (Nash) manifolds. Denote $D_Y^X := D_\phi := \phi^*(D_Y^*) \otimes D_X$.*

(iii) *Let $\mathcal{E} \rightarrow Y$ be a (Nash) bundle. Denote $\phi^*(\mathcal{E}) = \phi^*(\mathcal{E}) \otimes D_Y^X$.*

Remark 6.0.6. *If ϕ is a submersion then for all $y \in Y$ we have $D_Y^X|_{\phi^{-1}(y)} \cong D_{\phi^{-1}(y)}$.*

6.1. Tempered functions. Analyzing smooth representations from a geometric point of view we encounter two related technical difficulties:

- Most characters of F^\times are not Nash
- ψ is not Nash.

This makes the group action on the geometric objects that we consider to be not Nash. This could cause us to consider geometric objects outside the realm of Nash manifolds. In order to prevent that we introduce some technical notions, including the notion of G -tempered bundle which is, roughly speaking, a Nash bundle with a tempered G -action. The price of that is the need to make sure that all our constructions produce only Nash geometric objects.

Notation 6.1.1. *Let X be a Nash manifold. Denote by $\mathcal{T}(X)$ the space of tempered functions on X . For any Nash vector bundle \mathcal{E} over X we denote by $\mathcal{T}(X, \mathcal{E})$ the space of tempered functions of \mathcal{E} .*

Note that any unitary character of a real algebraic group G is a tempered function on G .

Definition 6.1.2. *Let X be a Nash manifold and let \mathcal{E} and \mathcal{E}' be Nash bundles over it. A morphism of bundles $\phi : \mathcal{E} \rightarrow \mathcal{E}'$ is called tempered if it corresponds to a tempered section of the bundle $\text{Hom}(\mathcal{E}, \mathcal{E}')$.*

Note that if $\phi : \mathcal{E} \rightarrow \mathcal{E}'$ is a tempered morphism of bundles, then the induced map $C^\infty(X, \mathcal{E}) \rightarrow C^\infty(X, \mathcal{E}')$ maps $\mathcal{S}(X, \mathcal{E})$ to $\mathcal{S}(X, \mathcal{E}')$ and $\mathcal{T}(X, \mathcal{E})$ to $\mathcal{T}(X, \mathcal{E}')$.

Definition 6.1.3. *Let a Nash group G act on a Nash manifold X . A **G -tempered bundle** \mathcal{E} over X is a Nash bundle \mathcal{E} with an equivariant structure $\phi : a^*(\mathcal{E}) \rightarrow p^*(\mathcal{E})$ (here $a : G \times X \rightarrow X$ is the action map and $p : G \times X \rightarrow X$ is the projection) that is a tempered morphism of bundles such that for any element L in the Lie algebra of G and any open (semi-algebraic) subset $U \subset X$ the derived map $a(L) : C^\infty(U, \mathcal{E}) \rightarrow C^\infty(U, \mathcal{E})$ preserves the sub-space of Nash sections of \mathcal{E} on U .*

Definition 6.1.4. *• We call a character χ of a Nash group G multiplicative if $\chi = \mu \circ \overline{\chi}$, where $\overline{\chi} : G \rightarrow F^\times$ is a homomorphism of Nash groups and $\mu : F^\times \rightarrow \mathbb{C}^\times$ is a character. Note that all multiplicative characters are tempered.*

- A multiplicative representation of G is a product of (finite-dimensional) Nash representation of G with a multiplicative character.
- Let a Nash group G act on a Nash manifold X . A G -tempered bundle \mathcal{E} over X is called **G -multiplicative bundle** if for any point $x \in X$, the fiber \mathcal{E}_x is a multiplicative representation of the stabilizer G_x .

7. PROOF OF EXACTNESS AND HAUSDORFFNESS (THEOREM 3.0.8)

We will prove Theorem 3.0.8 for the “untwisted” functor \tilde{E}_∞^k . Clearly, these versions are equivalent. The proof is based on the following theorem.

Theorem 7.0.1. *There exists a sequence of collections $\{C_n\}_{n=1}^\infty$ of topological Hausdorff \mathfrak{p}_n -representations such that*

- (1) *Any representation parabolically induced from a finite-dimensional representation of the torus lies in C_n .*
- (2) *For any $\pi \in C_n$, $\pi \otimes (-\psi)$ is \mathfrak{v}_n -acyclic (as a linear representation).*
- (3) *For any $\pi \in C_n$, we have $\pi_{\mathfrak{v}_n, \psi} \in C_{n-1}$.*

We will prove this theorem in §§7.1.

Corollary 7.0.2. *Any \mathfrak{p}_n -representation π of the class C_n is acyclic with respect to \tilde{E}_∞^k and $\tilde{E}_\infty^k(\pi)$ is Hausdorff, for any $0 < k \leq n$.*

This corollary follows from Theorem 7.0.1, using the following lemma that we will prove in §§9.3.

Lemma 7.0.3. *Let π be a representation of \mathfrak{p}_n . Suppose that π is $\tilde{\Phi}$ -acyclic and $\tilde{\Phi}(\pi)$ is \tilde{E}^k -acyclic. Then π is \tilde{E}^{k+1} -acyclic.*

Corollary 7.0.4. *Any representation π of G_n parabolically induced from a finite-dimensional representation of the torus is acyclic with respect to \tilde{E}_∞^k and $\tilde{E}_\infty^k(\pi)$ is Hausdorff, for any $0 < k \leq n$.*

Now we are ready to deduce Theorem 3.0.8. The deduction follows the lines of the proof of [CHM00, Lemma 8.4].

Proof of Theorem 3.0.8. (1) First of all, note that it is enough to show that for any $\pi \in \mathcal{M}_\infty(G_n)$, the $\mathfrak{u}^{n,k}$ -representation $\pi \otimes (-\psi_n^k)$ is acyclic.

We will prove by downward induction on l that $H^l(\mathfrak{u}_n^k, \pi \otimes (-\psi_n^k)) = 0$ for any $\pi \in \mathcal{M}_\infty(G_n)$. For $l > \dim \mathfrak{u}_n^k$ it follows from the Koszul complex (see §§9.2.1). Let us assume that the statement holds for $l+1$ and prove it for l . By the Casselman subrepresentation theorem (Theorem 2.2.6), one can embed π into a representation I parabolically induced from a finite dimensional representation of the torus. Consider the short exact sequence

$$0 \rightarrow \pi \rightarrow I \rightarrow I/\pi \rightarrow 0.$$

By Corollary 2.2.5, $I/\pi \in \mathcal{M}_\infty(G_n)$ and thus, by the induction hypothesis, $H^{l+1}(\mathfrak{u}_n^k, (I/\pi) \otimes (-\psi_n^k)) = 0$. By Corollary 7.0.4, $H^i(\mathfrak{u}_n^k, I \otimes (-\psi_n^k)) = 0$ for all $i > 0$. Thus, $H^l(\mathfrak{u}_n^k, \pi \otimes (-\psi_n^k)) = H^{l+1}(\mathfrak{u}_n^k, (I/\pi) \otimes (-\psi_n^k)) = 0$.

(2) By the Casselman subrepresentation theorem (Theorem 2.2.6), one can embed π into a representation I parabolically induced from a finite dimensional representation of the torus. By (1), we have an embedding $\tilde{E}_\infty^k(\pi) \hookrightarrow \tilde{E}_\infty^k(I)$. By Corollary 7.0.4, $\tilde{E}_\infty^k(I)$ is Hausdorff and hence $\tilde{E}_\infty^k(\pi)$ is Hausdorff. \square

7.1. Good \mathfrak{p}_n -representations and the analytic key lemma. The following lemma will play a key role in this and the next section.

Lemma 7.1.1 (The analytic key lemma). *Let T be a Nash linear group, i.e. a Nash subgroup of $GL_k(\mathbb{R})$ for some k , and let $R := P_n \times T$ and $R' := G_{n-1} \times T$. Let $Q < R$ be a Nash subgroup and let $X := R/Q$ and $X' := V_n \setminus X$. Note that $X' = R'/Q'$, where $Q' = Q/(Q \cap V_n)$. Let $X_0 = \{x \in X : \psi|_{(V_n)_x} = 1\}$. Let X'_0 be the image of X_0 in X' . Let \mathcal{E}' be an R' -tempered bundle on X' and $\mathcal{E} := p_{X'}^*(\mathcal{E}')$ be its pullback to X . Then*

- (i) $H_i(\mathfrak{v}_n, \mathcal{S}(X, \mathcal{E}) \otimes (-\psi)) = 0$ for any $i > 0$.
- (ii) X'_0 is smooth
- (iii) $H_0(\mathfrak{v}_n, \mathcal{S}(X, \mathcal{E}) \otimes (-\psi)) \cong \mathcal{S}(X'_0, \mathcal{E}|_{X'_0})$ as representations of \mathfrak{p}_{n-1} .

We will prove this lemma in §10.

Definition 7.1.2. Let $B_n < G_n$ denote the standard (upper-triangular) Borel subgroup. In the situation of the above lemma, assuming that $(B_n \cap P_n) \times T$ has finite number of orbits on X , and that \mathcal{E}' is R' -multiplicative we call the representation $\mathcal{S}(X, \mathcal{E})$ a geometric representation of \mathfrak{p}_n .

Definition 7.1.3. We define a good extension of nuclear Fréchet spaces to be the following data:

- (1) a nuclear Fréchet space W
- (2) a countable descending filtration $F^i(W)$ by closed subspaces
- (3) a sequence of nuclear Fréchet spaces W_i
- (4) isomorphisms $\phi_i : F^{i+1}(W)/F^i(W) \rightarrow W_i$

such that the natural map $W \rightarrow \lim_{\leftarrow} W/F^i(W)$ is an isomorphism of (non-topological) linear spaces.

In this situation we will also say that W is a good extension of W_i . If a Lie algebra \mathfrak{g} acts on W and on W_i , preserving $F^i(W)$ and commuting with ϕ we will say that this good extension is \mathfrak{g} -invariant.

Definition 7.1.4. We define the collection of good representations of \mathfrak{p}_n to be the smallest collection that includes the geometric representations and is closed with respect to \mathfrak{p}_n -invariant good extensions.

We will prove the following concretization of Theorem 7.0.1:

Theorem 7.1.5.

- (1) The representations of G_n parabolically induced from a finite dimensional representation of the torus are good representations of \mathfrak{p}_n .
- (2) For any good representation π of \mathfrak{p}_n , $\pi \otimes (-\psi)$ is \mathfrak{v}_n -acyclic (as a linear representation).
- (3) For any good representation π of \mathfrak{p}_n , the representation $\pi_{\mathfrak{v}_n, \psi}$ is good representation of \mathfrak{p}_{n-1} .

For the proof we will need some lemmas.

Lemma 7.1.6. Let T be a Nash linear group, and let $R := P_n \times T$ and $R' := G_{n-1} \times T$. Let Y be an R -Nash manifold. Let $X \subset Y$ be an R -invariant (locally closed) Nash submanifold such that $(B_n \cap P_n) \times T$ has finite number of orbits on X . Let \mathcal{E} be an R -multiplicative bundle on Y . Then the representation $\mathcal{S}_Y(X, \mathcal{E})$ is good.

We will prove this lemma in §§9.7.

Lemma 7.1.7. Let G be a Nash group and H be a closed Nash subgroup. Let χ be a multiplicative character of H . Let K be a closed Nash subgroup of G such that $KH = G$ and $\chi|_{K \cap H}$ is a Nash character. Let $\mathcal{E} = G \times_H \chi$ be the smooth bundle on G/H obtained by inducing the character χ . Then \mathcal{E} admits a structure of a G -multiplicative bundle.

We will prove this lemma in §§9.8.

Corollary 7.1.8. Let χ be a character of the torus $T_n < B_n < G_n$, continued trivially to B_n . Let $\pi = \text{Ind}_{B_n}^{G_n}(\chi)$. Then there exists a G_n -multiplicative bundle \mathcal{E} on G_n/B_n such that $\pi = \mathcal{S}(G_n/B_n, \mathcal{E})$.

Lemma 7.1.9. Let \mathfrak{g} be a Lie algebra. Let (π, F, π_i) be a \mathfrak{g} -invariant good extension. Suppose that π_i are \mathfrak{g} -acyclic and $H_0(\mathfrak{g}, \pi_i)$ is Hausdorff for any i . Then π is acyclic and $H_0(\mathfrak{g}, \pi)$ together with the induced filtration $F^i(H_0(\mathfrak{g}, \pi)) = F^i(\pi)/(\mathfrak{g}\pi \cap F^i(\pi))$ and the natural morphisms $F^i(H_0(\mathfrak{g}, \pi))/F^{i+1}(H_0(\mathfrak{g}, \pi)) \rightarrow H_0(\mathfrak{g}, \pi_i)$ form a good extension. In particular, $H_0(\mathfrak{g}, \pi)$ is Hausdorff.

We will prove this lemma in §§9.5.

Proof of Theorem 7.1.5.

- (1) Let σ be a finite dimensional representation of the torus and let $\pi \in \mathcal{M}_\infty(G_n)$ be its parabolic induction. Without loss of generality, by the definition of goodness, we can assume that σ is a character. In this case π is good by Corollary 7.1.8 and Lemma 7.1.6.
- (2) The analytic key lemma (Lemma 7.1.1) implies that $\pi \otimes (-\psi)$ is acyclic for any geometric representation π . Hence Lemma 7.1.9 implies by induction that $\pi \otimes (-\psi)$ is acyclic for any good representation π .

(3) If $\pi = \mathcal{S}(X, \mathcal{E})$ is a geometric representation then by the analytic key lemma (Lemma 7.1.1), $\pi_{\mathfrak{v}_n, \psi} \cong \mathcal{S}(X'_0, \mathcal{E}'|_{X'_0})$, where \mathcal{E}' and X'_0 are as in the analytic key lemma. Let \tilde{T} denote the product of T with the center of G_{n-1} . Clearly, \tilde{T} acts on X'_0 and $B_{n-1} \times T = (B_{n-1} \cap P_{n-1}) \times \tilde{T}$ has a finite number of orbits on X'_0 . Thus, by Lemma 7.1.6, $\pi_{\mathfrak{v}_n, \psi}$ is good.

Now let π be a good representation. We can assume by induction that π is a good extension of π_i , where π_i are good and $(\pi_i)_{\mathfrak{v}_n, \psi}$ are good. Then Lemma 7.1.9 implies that $\pi_{\mathfrak{v}_n, \psi}$ is a good extension of $(\pi_i)_{\mathfrak{v}_n, \psi}$ and hence is good.

□

For the proof of Theorem 3.0.13 we will need the following corollary of Lemma 7.1.9 and Theorem 7.1.5.

Corollary 7.1.10. *Let (π, F, π_i) be a \mathfrak{p}_n -invariant good extension of good representations of \mathfrak{p}_n . Then $E^k(\pi)$ together with the induced filtration defined by the images of $E^k(F^i(\pi))$ and the natural morphisms $F^i(E^k(\pi))/F^{i+1}(E^k(\pi)) \rightarrow E^k(\pi_i)$ form a good extension.*

8. HIGHEST DERIVATIVE OF MONOMIAL REPRESENTATIONS (PROOF OF THEOREM 3.0.13)

We first sketch the proof for the case of product of two characters. We believe that using an appropriate notion of Fréchet bundles it will be possible to upgrade this proof to the case of product of two representations, which by induction will give a proof of Conjecture 3.1.1 (and thus also Theorem 3.0.13).

Since we do not currently have a proof for a product of two representations, in §§8.3 - 8.5 we prove Theorem 3.0.13 directly for a product of k characters.

8.1. Sketch of proof for the case $k = 2$. First, note that there exists a G_n -multiplicative line bundle \mathcal{E} over $G_n/P_{(n_1, n_2)}$ such that $\chi_1 \times \chi_2 = \mathcal{S}(G_n/P_{(n_1, n_2)}, \mathcal{E})$ and the action of $P_{(n_1, n_2)}$ on the fiber is given by trivial extension of $\chi_1 \otimes \chi_2$, twisted by a power of the determinant.

Let $w_n \in G_n$ denote the longest Weyl group element. Let $x_1, x_2 \in X$ be the classes of 1 and w_n in correspondence and let \mathcal{O}_1 and \mathcal{O}_2 be their P_n -orbits. Note that $X = \mathcal{O}_1 \cup \mathcal{O}_2$. Thus, by Corollary 6.0.4, we have a short exact sequence

$$0 \rightarrow \mathcal{S}(\mathcal{O}_2, \mathcal{E}) \rightarrow \mathcal{S}(X, \mathcal{E}) \rightarrow \mathcal{S}_X(\mathcal{O}_1, \mathcal{E}) \rightarrow 0.$$

By Lemma 7.1.6, $\mathcal{S}(\mathcal{O}_2, \mathcal{E})$ and $\mathcal{S}_X(\mathcal{O}_1, \mathcal{E})$ are good. By Theorem 7.1.5, $\mathcal{S}(\mathcal{O}_2, \mathcal{E})$ and $\mathcal{S}_X(\mathcal{O}_1, \mathcal{E})$ are acyclic with respect to Φ and thus we have a short exact sequence

$$0 \rightarrow \Phi(\mathcal{S}(\mathcal{O}_2, \mathcal{E})) \rightarrow \Phi(\mathcal{S}(X, \mathcal{E})) \rightarrow \Phi(\mathcal{S}_X(\mathcal{O}_1, \mathcal{E})) \rightarrow 0.$$

The proposition follows from the following 2 statements:

- (1) $\Phi(\mathcal{S}_X(\mathcal{O}_1, \mathcal{E})) = 0$
- (2) $\Phi(\mathcal{S}(\mathcal{O}_2, \mathcal{E})) = (\pi_1)|_{G_{n_1-1}} \times (\pi_2)|_{G_{n_2-1}}$.

Let us first show (1). Using the results of §7 and a version of the Borel lemma (Lemma 9.6.17), we reduce to the statement $\Phi(\mathcal{S}(\mathcal{O}_1, \mathcal{E}')) = 0$ for any P_n -multiplicative equivariant bundle \mathcal{E}' . Let $p : \mathcal{O}_1 \rightarrow V_n \setminus \mathcal{O}_1$ be the natural projection. Using the nilpotency of the action of V_n on \mathcal{E}' we reduce to the case $\mathcal{E}' = p^*(\mathcal{E}'')$. This case follows by the analytic key lemma (Lemma 7.1.1) from the inclusion $P_{x_1} = P_n \cap P_{(n_1, n_2)} \supset V_n$.

The proof of (2) is a computation based on the analytic key lemma (Lemma 7.1.1).

8.2. Structure of the proof. Let $\lambda = (n_1, \dots, n_k)$ be a composition of n . The monomial representation is isomorphic to $\mathcal{S}(G_n/P_\lambda, \mathcal{E})$ for some G_n -multiplicative line bundle \mathcal{E} over $X := G_n/P_\lambda$. Let $\mathcal{O}_1, \dots, \mathcal{O}_k$ be the P_n -orbits on G_n/P_λ , where \mathcal{O}_1 is the open orbit. We reduce the problem to the following two tasks:

- to compute $E^k(\mathcal{S}(\mathcal{O}_1, \mathcal{E}))$
- to show that $E^k(\mathcal{S}_X(\mathcal{O}_i, \mathcal{E}'))$ for any P_n -multiplicative equivariant bundle \mathcal{E}' on X any $2 \leq i \leq k$.

We do both tasks by induction using the analytic key lemma (Lemma 7.1.1) and the fact that the P_{n-1} -orbits on the geometric quotient $V_n \setminus \mathcal{O}_i$ looks exactly like \mathcal{O}_j when n is replaced by $n-1$.

In §§8.3 we study the geometry of P_n -orbits on X . In §§8.4 we reformulate the above tasks in explicit lemmas, that will be proven in §§9.9. In §§8.5 we prove Theorem 3.0.13 using those lemmas.

8.3. Geometry of P_n -orbits on flag varieties. Let $\lambda = (n_1, \dots, n_k)$ be a composition of n . In this subsection we describe the orbits of P_n on G_n/P_λ . Note that the scalar matrices act trivially on G/P_λ and thus P_n orbits coincide with $P_{n-1,1}$ -orbits. By Bruhat theory, $P_{(n-1,1)} \setminus G_n/P_\lambda$ can be identified with the set of double cosets of the symmetric group permuting the set $\{1, \dots, n\}$ by subgroups corresponding to $P_{(n-1,1)}$ and P_λ respectively. The first subgroup is the stabilizer of the point n , and the second is the subgroup of all permutations that preserve the segment $[n_{i-1}+1, n_i]$ for each i . Thus, the double quotient consists of k elements. Now we want to find a representative in G_n for each double coset. We can choose all of them to be powers of the standard cyclic permutation matrix. This discussion is formalized in the following notation and lemma.

Notation 8.3.1.

- Let $c \in G_n$ denote the standard cyclic permutation matrix, i.e. $ce_i = e_{i+1}$ for $i < n$ and $ce_n = e_1$, where e_i are basis vectors.
- $m_\lambda^i := \sum_{l=1}^i n_l$. In particular, $m_\lambda^k = n$.
- For $1 \leq i \leq k$, let $w_\lambda^i := c^{n-m_\lambda^{k-i+1}} \in G_n$
- $P_\lambda^i := w_\lambda^i P_\lambda (w_\lambda^i)^{-1}$.
- Let x_λ^i be the class of w_λ^i in G_n/P_λ and $\mathcal{O}_\lambda^i := P_n x_\lambda^i$.
- $Q_\lambda^i := P_\lambda^i \cap P_n$.

See Appendix C for examples of the objects described in this and the following notations.

Lemma 8.3.2.

- (i) The stabilizer of x_λ^i is Q_λ^i .
- (ii) \mathcal{O}_λ^i are all distinct.
- (iii) $G_n/P_\lambda = \bigcup_{i=1}^k \mathcal{O}_\lambda^i$
- (iv) For any $l \leq k$, $\bigcup_{i=1}^l \mathcal{O}_\lambda^i$ is closed in G_n/P_λ .

We will also be interested in the representatives of P_n -orbits on G_n/P_λ^i .

Notation 8.3.3. Let $\lambda = (n_1, \dots, n_k)$ be a composition of n . For $1 \leq i, j \leq k$,

- Let $w_\lambda^{ij} := (w_\lambda^i)^{-1} w_\lambda^j \in G_n$
- Let x_λ^{ij} be the class of w_λ^{ij} in G_n/P_λ^i and $\mathcal{O}_\lambda^{ij} := P_n x_\lambda^{ij}$.

Corollary 8.3.4. Fix $i \leq k$. Then

- (i) The stabilizer of x_λ^{ij} is Q_λ^j .
- (ii) \mathcal{O}_λ^{ij} are all distinct.
- (iii) $G_n/P_\lambda^i = \bigcup_{j=1}^k \mathcal{O}_\lambda^{ij}$
- (iv) For any $l \leq k$, $\bigcup_{j=1}^l \mathcal{O}_\lambda^{ij}$ is closed in G_n/P_λ^i .

$\{x_\lambda^{ij}\}_{j=1}^k$ is a full system of representatives for P_n -orbits in G_n/P_λ^i , and the stabilizer of x_λ^{ij} is Q_λ^j .

Notation 8.3.5. For any $1 \leq i \leq k$, denote $\lambda_i^- := (n_1, \dots, n_{i-1}, n_i - 1, n_{i+1}, \dots, n_k)$.

The following lemma is a straightforward computation (see Appendix C for a particular example.).

Lemma 8.3.6. Let $\lambda = (n_1, \dots, n_k)$ be a composition of n . Let $X := P/Q_\lambda^i$ and $Z := (V_n) \setminus X$ be the geometric quotient. Note that $Z = G_{n-1}/P_{\lambda'}^i$, where $\lambda' = \lambda_{k-i+1}^-$.

Let $L = Q_\lambda^i \cap V_n$ and $\tilde{Z}_0 := \{g \in G_{n-1} : \psi(gL) = 1\}$. Note that \tilde{Z}_0 is right-invariant with respect to $P_{\lambda'}^i$. Let $Z_0 := \tilde{Z}_0 / P_{\lambda'}^i \subset Z$.

Then $x_{\lambda'}^{ij} \in Z_0$ if and only if $1 \leq j < i$.

8.4. Derivatives of quasi-regular representations on P_n -orbits on flag varieties. The proof of Theorem 3.0.13 is based on two lemmas that we formulate in this subsection and prove in §§9.9.

Lemma 8.4.1. *Let $\lambda = (n_1, \dots, n_k)$ be a composition of n . Let Y be a Nash P_n -manifold and \mathcal{E} be a P_n -multiplicative bundle on Y . Let $x_0 \in Y$ be a point with stabilizer Q_λ^i and let $X := P_n x_0$. Let $\pi := \mathcal{S}_Y(X, \mathcal{E})$. Then $E^{i+1}(\pi) = 0$.*

Notation 8.4.2. • Denote by \mathfrak{X} the set of all characters of F^\times .

- Let $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathfrak{X}^k$. Denote by $\bar{\xi}_{\lambda, \alpha}$ the character of L_λ defined by

$$\bar{\xi}_{\lambda, \alpha}(g_1, \dots, g_k) = \prod_{i=1}^k (\alpha_i(\det(g_i)))$$

and by $\xi_{\lambda, \alpha}$ the character of L_λ defined by

$$\xi_{\lambda, \alpha}(g_1, \dots, g_k) = \bar{\xi}_{\lambda, \alpha} \Delta_{P_\lambda}^{1/2}|_{L_\lambda} = \prod_{i=1}^k (\alpha_i(\det(g_i)) |\det(g_i)|^{(n-m_\lambda^i - m_\lambda^{i-1})/2}).$$

- Let $\chi_{\lambda, \alpha}$ and $\bar{\chi}_{\lambda, \alpha}$ be the extensions of $\xi_{\lambda, \alpha}$ and $\bar{\xi}_{\lambda, \alpha}$ to P_λ .
- Denote by $\chi_{\lambda, \alpha}^i$ and $\bar{\chi}_{\lambda, \alpha}^i$ the characters of P_λ^i defined by $\chi_{\lambda, \alpha}^i(g) = \chi_{\lambda, \alpha}((w_\lambda^i)^{-1} g w_\lambda^i)$ and $\bar{\chi}_{\lambda, \alpha}^i(g) = \bar{\chi}_{\lambda, \alpha}((w_\lambda^i)^{-1} g w_\lambda^i)$.

Lemma 8.4.3. *Let $\lambda = (n_1, \dots, n_k)$ be a composition of n . Let $X := P_n/Q_\lambda^i$. Let \mathcal{E} be a P_n -multiplicative line bundle on X . Let $\pi := \mathcal{S}(X, \mathcal{E})$. Let $x_0 \in X$ be the class of the trivial element of P_n . Consider $\mathcal{E}|_{x_0}$ as a character of Q_λ^i . Suppose that this character is the restriction of $\chi_{\lambda, \alpha}^i$ to Q_λ^i , for some $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathfrak{X}^k$. Then*

$$E^i(\pi)|_{G_{n-i}} = \chi_{(n_1), (\alpha_1)} |\det|^{-1/2} \times \cdots \times \chi_{(n_{k-i}), (\alpha_{k-i})} |\det|^{-1/2} \times \chi_{(n_{k-i+1}-1), (\alpha_{k-i+1})} \times \cdots \times \chi_{(n_k-1), (\alpha_k)}$$

8.5. Proof of Theorem 3.0.13.

Lemma 8.5.1. *Let $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathfrak{X}^k$. There exists a G_n -multiplicative line bundle \mathcal{E} over G/P_λ such that $\chi_{(n_1), (\alpha_1)} \times \cdots \times \chi_{(n_k), (\alpha_k)} = \mathcal{S}(G/P_\lambda, \mathcal{E})$ and the action of P_λ on the fiber $\mathcal{E}_{x_\lambda^1}$ is given by the character $\chi_{\lambda, \alpha}$.*

This lemma follows from Lemma 7.1.7.

Proof of Theorem 3.0.13. Let $X := G_n/P_\lambda$ and let \mathcal{E} be the G_n -multiplicative equivariant bundle given by Lemma 8.5.1. We have to show that $E^k(\mathcal{S}(X, \mathcal{E})) = \chi_{(n_1-1), (\alpha_1)} \times \cdots \times \chi_{(n_k-1), (\alpha_k)}$. Recall that by Lemma 8.3.2, $G_n/P_\lambda = \bigcup_{i=1}^k \mathcal{O}_\lambda^i$. Thus, by Corollary 6.0.4 there is a finite filtration on $\mathcal{S}(X, \mathcal{E})$ such that $Gr^i(\mathcal{S}(X, \mathcal{E})) = \mathcal{S}_X(\mathcal{O}_\lambda^i, \mathcal{E})$. By Lemma 7.1.6, each $\mathcal{S}_X(\mathcal{O}_\lambda^i, \mathcal{E})$ is a good representation of \mathfrak{p}_n and thus, by Theorem 7.1.5, $\mathcal{S}_X(\mathcal{O}_\lambda^i, \mathcal{E})$ is E^k -acyclic. Thus, $E^k(\mathcal{S}(X, \mathcal{E}))$ has a filtration such that $Gr^i(E^k(\mathcal{S}(X, \mathcal{E}))) = E^k(\mathcal{S}_X(\mathcal{O}_\lambda^i, \mathcal{E}))$. By Lemma 8.4.1, $E^k(\mathcal{S}_X(\mathcal{O}_\lambda^i, \mathcal{E})) = 0$ for all $i < k$. By Lemma 8.4.3,

$$E^k(\mathcal{S}(\mathcal{O}_\lambda^i, \mathcal{E}))|_{G_{n-k}} = \chi_{(n_1-1), (\alpha_1)} \times \cdots \times \chi_{(n_k-1), (\alpha_k)}$$

Thus,

$$E^k(\mathcal{S}(X, \mathcal{E}))|_{G_{n-k}} = E^k(\mathcal{S}(\mathcal{O}_\lambda^k, \mathcal{E}))|_{G_{n-k}} = \chi_{(n_1-1), (\alpha_1)} \times \cdots \times \chi_{(n_k-1), (\alpha_k)}$$

□

Remark 8.5.2. Note that this proof gives a description of $E^l(\chi_1 \times \cdots \times \chi_k)$ for an arbitrary l . More precisely, we get a filtration on this space and a description of each quotient. However, this is an infinite filtration, indexed by a rather complicated ordered set.

In fact, we have a recipe to compute $\Phi(\mathcal{S}(P/Q_\lambda^i, \mathcal{E}))$ for any multiplicative bundle \mathcal{E} . First, we use Lemma 9.7.1 to reduce to the case when \mathcal{E} is the pullback of a bundle \mathcal{E}' on the geometric quotient $P_n/Q_\lambda^i V_n$. Then the analytic key lemma (Lemma 7.1.1) says $\Phi(\mathcal{S}(P/Q_\lambda^i, \mathcal{E})) = \mathcal{S}(X'_0, \mathcal{E}')$, where $X'_0 \subset P_n/Q_\lambda^i V_n$. The set X'_0 is described in Lemma 8.3.6. This gives us a filtration on $\mathcal{S}(X'_0, \mathcal{E}')$ such that the associated graded pieces are of the type $\mathcal{S}(P_{n-1}/Q_{\lambda_i^-}^j, \mathcal{E}_{jk})$ where $1 \leq j < i$ and $k \geq 0$.

Since the orbits of P_n on X/P_λ are P_n/Q_λ^i , we can use this recipe in order to describe the value of the functor Φ on products of finite-dimensional representations and then, proceeding inductively, we can describe the functor E^i on such representations.

9. PROOFS OF SOME TECHNICAL LEMMAS

9.1. Finite dimensional dense subspaces (Proof of Lemma 3.0.10).

Lemma 9.1.1. *Any locally convex Hausdorff topology on a finite dimensional vector space is equivalent to the standard topology.*

Proof. Since W is finite dimensional, we have a linear isomorphism $\phi : \mathbb{R}^n \rightarrow W$, which is continuous. Thus, the topology on W is not stronger than the classical one. Hence it is left to prove that for any $r > 0$, $\phi(B(0, r))$ contains an open neighborhood of 0, where $B(0, r)$ denotes the open ball with radius r and center at the origin.

Since W is Hausdorff, for any compact $C \subset \mathbb{R}^n$, $\phi(C)$ is closed. Consider $\phi(B(0, r) \cup (\mathbb{R}^n \setminus \overline{B(0, 2r)}))$. It is an open neighborhood of 0, hence it contains an open convex neighborhood of 0 which in turn must be contained in $\phi(B(0, r))$. \square

We immediately obtain

Corollary 9.1.2. *If a locally convex Hausdorff topological vector space W has a dense finite dimensional subspace then W is finite dimensional.*

9.2. Preliminaries on homological algebra.

Definition 9.2.1. *Let \mathcal{C} be an abelian category. A family of objects $\mathcal{A} \subset Ob(\mathcal{C})$ is called a **generating family** if for any object $X \in Ob(\mathcal{C})$ there exists an object $Y \in \mathcal{A}$ and an epimorphism $Y \twoheadrightarrow X$.*

Definition 9.2.2 ([GM88], III.6.3). *Let \mathcal{C} and \mathcal{D} be abelian categories and $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ be a right-exact additive functor. A family of objects $\mathcal{A} \subset Ob(\mathcal{C})$ is called **\mathcal{F} -adapted** if it is generating, closed under direct sums and for any acyclic complex $\cdots \rightarrow A_3 \rightarrow A_2 \rightarrow A_1 \rightarrow 0$ with $A_i \in \mathcal{A}$, the complex $\cdots \rightarrow \mathcal{F}(A_3) \rightarrow \mathcal{F}(A_2) \rightarrow \mathcal{F}(A_1) \rightarrow 0$ is also acyclic.*

For example, a generating, closed under direct sums system consisting of projective objects is \mathcal{F} -adapted for any right-exact functor \mathcal{F} . For a Lie algebra, the system of free \mathfrak{g} -modules (i.e. direct sums of copies of $U(\mathfrak{g})$) is an example of such system.

The following results are well-known.

Theorem 9.2.3. *Let \mathcal{C} and \mathcal{D} be abelian categories and $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ be a right-exact additive functor. Suppose that there exists an \mathcal{F} -adapted family $\mathcal{A} \subset Ob(\mathcal{C})$. Then \mathcal{F} has derived functors.*

Remark 9.2.4. *Note that if the functor \mathcal{F} has derived functors, and \mathcal{A} is a generating class of acyclic objects then \mathcal{A} is \mathcal{F} -adapted.*

Lemma 9.2.5. *Let \mathcal{A} , \mathcal{B} and \mathcal{C} be abelian categories. Let $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ and $\mathcal{G} : \mathcal{B} \rightarrow \mathcal{C}$ be right-exact additive functors. Suppose that both \mathcal{F} and \mathcal{G} have derived functors.*

(i) *Suppose that \mathcal{F} is exact. Suppose also that there exists a class $\mathcal{E} \subset Ob(\mathcal{A})$ which is $\mathcal{G} \circ \mathcal{F}$ -adapted and such that $\mathcal{F}(X)$ is \mathcal{G} -acyclic for any $X \in \mathcal{E}$. Then the functors $L^i(\mathcal{G} \circ \mathcal{F})$ and $L^i\mathcal{G} \circ \mathcal{F}$ are isomorphic.*

(ii) *Suppose that there exists a class $\mathcal{E} \subset Ob(\mathcal{A})$ which is $\mathcal{G} \circ \mathcal{F}$ -adapted and \mathcal{F} -adapted and such that $\mathcal{F}(X)$ is \mathcal{G} -acyclic for any $X \in \mathcal{E}$. Let $Y \in \mathcal{A}$ be an \mathcal{F} -acyclic object. Then $L^i(\mathcal{G} \circ \mathcal{F})(Y)$ is (naturally) isomorphic to $L^i\mathcal{G}(\mathcal{F}(Y))$.*

(iii) *Suppose that \mathcal{G} is exact. Suppose that there exists a class $\mathcal{E} \subset Ob(\mathcal{A})$ which is $\mathcal{G} \circ \mathcal{F}$ -adapted and \mathcal{F} -adapted. Then the functors $L^i(\mathcal{G} \circ \mathcal{F})$ and $\mathcal{G} \circ L^i\mathcal{F}$ are isomorphic.*

9.2.1. Lie algebra homology. Let \mathfrak{g} be a Lie algebra and $\pi \in \mathcal{M}(\mathfrak{g})$. The homology of \mathfrak{g} with coefficients in π are defined by $H_i(\mathfrak{g}, \pi) := L^i\mathcal{C}(\pi)$, where $\mathcal{C} : \mathcal{M}(\mathfrak{g}) \rightarrow Vect$ is the functor of coinvariants and $L^i\mathcal{C}$ denotes the derived functor. The homology $H_i(\mathfrak{g}, \pi)$ is equal to the i -th homology of the Koszul complex.

$$(17) \quad 0 \leftarrow \pi \leftarrow \mathfrak{g} \otimes \pi \leftarrow \cdots \leftarrow \Lambda^n(\mathfrak{g}) \otimes \pi \leftarrow 0.$$

The following lemmas are standard and can be easily deduced from Lemma 9.2.5.

Lemma 9.2.6. *Let $\mathfrak{h} \subset \mathfrak{g}_1 \subset \mathfrak{g}$ be Lie algebras. Let ψ be a character of \mathfrak{h} . Suppose that \mathfrak{g}_1 stabilizes (\mathfrak{h}, ψ) . Let $\mathcal{F} : \mathcal{M}(\mathfrak{g}) \rightarrow \mathcal{M}(\mathfrak{g}_1)$ be the functor defined by $\mathcal{F}(\pi) = \pi_{\mathfrak{h}, \psi}$. Then $L^i(\mathcal{F})(\pi) = H_i(\mathfrak{h}, \pi \otimes (-\psi))$.*

Lemma 9.2.7. *Let \mathfrak{g} be a Lie algebra and \mathfrak{i} be its Lie ideal. Let π be a representation of \mathfrak{g} . Suppose that π is \mathfrak{i} -acyclic and $H_0(\mathfrak{i}, \pi)$ is $\mathfrak{g}/\mathfrak{i}$ -acyclic. Then π is \mathfrak{g} -acyclic.*

The last lemma is in fact a special case of the Hochschild-Serre spectral sequence.

9.3. Acyclicity with respect to composition of derivatives (Proof of Lemma 7.0.3). By Lemma 9.2.5(ii), it is enough to show that there exists a class of representations of \mathfrak{p}_n that is $\tilde{\Phi}$ -adapted, \tilde{E}^{k+1} -adapted, and such that for every object τ in the class, $\tilde{\Phi}(\tau)$ is \tilde{E}^k -acyclic. We claim that the class of free $U(\mathfrak{p}_n)$ -modules satisfies this conditions.

For this it is enough to show that $U(\mathfrak{p}_n)$ is \tilde{E}^{k+1} and $\tilde{\Phi}$ acyclic and $\tilde{\Phi}(U(\mathfrak{p}_n))$ is \tilde{E}^k acyclic. By Lemma 9.2.6 it is equivalent to showing that

$$H_i(\mathfrak{u}_n^{k+1}, U(\mathfrak{p}_n) \otimes (-\psi_n^{k+1})) = H_i(\mathfrak{v}_n, U(\mathfrak{p}_n) \otimes (-\psi_n)) = H_i(\mathfrak{u}_{n-1}^k, U(\mathfrak{p}_n)_{\mathfrak{v}_n, \psi_n} \otimes (-\psi_{n-1}^k)) = 0,$$

for any $i > 0$. This follows from the fact that $U(\mathfrak{p}_n) \otimes (-\psi_n^{k+1})$ is free \mathfrak{u}_n^{k+1} module, $U(\mathfrak{p}_n) \otimes (-\psi_n)$ is free \mathfrak{v}_n -module and $U(\mathfrak{p}_n)_{\mathfrak{v}_n, \psi_n} \otimes (-\psi_{n-1}^k)$ is free \mathfrak{u}_{n-1}^k -module. This is immediate by the PBW theorem.

9.4. Preliminaries on topological linear spaces. We will need some classical facts from the theory of nuclear Fréchet spaces. A good exposition on nuclear Fréchet spaces can be found in [CHM00, Appendix A].

Definition 9.4.1. *We call a complex of topological vector spaces **admissible** if all its differentials have closed images.*

Proposition 9.4.2 (see e.g. [CHM00], Appendix A).

- (1) *Let V be a nuclear Fréchet space and W be a closed subspace. Then both W and V/W are nuclear Fréchet spaces.*
- (2) *Let*

$$\mathcal{C} : 0 \rightarrow C_1 \rightarrow \cdots \rightarrow C_n \rightarrow 0$$

be an admissible complex of nuclear Fréchet spaces. Then $H^i(\mathcal{C})$ are nuclear Fréchet spaces.

- (3) *Let*

$$\mathcal{C} : 0 \rightarrow C_1 \rightarrow \cdots \rightarrow C_n \rightarrow 0$$

be an admissible complex of nuclear Fréchet spaces. Then the complex \mathcal{C}^ is also admissible and $H_i(\mathcal{C}^*) \cong H^i(\mathcal{C})^*$.*

Corollary 9.4.3. *Let \mathfrak{g} be a (finite dimensional) Lie algebra and π be its nuclear Fréchet representation (i.e. a representation in a nuclear Fréchet space such that the action map $\mathfrak{g} \otimes \pi \rightarrow \pi$ is continuous). Suppose that $H_i(\mathfrak{g}, \pi)$ are Hausdorff for all $i \geq 0$. Then $H_i(\mathfrak{g}, \pi)$ are nuclear Fréchet and $H^i(\mathfrak{g}, \pi^*) = H_i(\mathfrak{g}, \pi)^*$.*

Proposition 9.4.4 (see e.g. [CHM00], Appendix A).

Let $0 \rightarrow V \rightarrow W \rightarrow U \rightarrow 0$ be an exact sequence of nuclear Fréchet spaces. Suppose that the embedding $V \rightarrow W$ is closed. Let L be a nuclear Fréchet space. Then the sequence $0 \rightarrow V \hat{\otimes} L \rightarrow W \hat{\otimes} L \rightarrow U \hat{\otimes} L \rightarrow 0$ is exact and the embedding $V \hat{\otimes} L \rightarrow W \hat{\otimes} L$ is closed.

9.5. Co-invariants of good extensions (Proof of Lemma 7.1.9). Let us first prove the following special case.

Lemma 9.5.1. *Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be a short exact sequence of nuclear Fréchet representations of a Lie algebra \mathfrak{g} . Suppose that L and N are acyclic and that $H_0(\mathfrak{g}, L)$ and $H_0(\mathfrak{g}, N)$ are Hausdorff.*

Then M is acyclic, $H_0(\mathfrak{g}, M)$ is Hausdorff, and $0 \rightarrow H_0(\mathfrak{g}, L) \rightarrow H_0(\mathfrak{g}, M) \rightarrow H_0(\mathfrak{g}, N) \rightarrow 0$ is a short exact sequence.

Proof. The long exact sequence implies that M is acyclic and $0 \rightarrow H_0(\mathfrak{g}, L) \rightarrow H_0(\mathfrak{g}, M) \rightarrow H_0(\mathfrak{g}, N) \rightarrow 0$ is a short exact sequence.

By Proposition 9.4.2(3) $0 \rightarrow N^* \rightarrow M^* \rightarrow L^*$ is exact, by Corollary 9.4.3 L^* is \mathfrak{g} -coacyclic (i.e. $H^i(\mathfrak{g}, L^*) = 0$ for $i > 0$) and hence $0 \rightarrow (N^*)^{\mathfrak{g}} \rightarrow (M^*)^{\mathfrak{g}} \rightarrow (L^*)^{\mathfrak{g}} \rightarrow 0$ is exact. We have to show that $\mathfrak{g}M$ is closed in M . For that it is enough to show that for any $x \in M$, if for any $f \in (M^*)^{\mathfrak{g}}$ we have $f(x) = 0$ then $x \in \mathfrak{g}M$. Let y be the image of x in N . We know that any $f \in (N^*)^{\mathfrak{g}}$ vanishes on y , and hence $y \in \mathfrak{g}N$. Hence $\exists x' \in \mathfrak{g}M$ such that $x - x' \in L$. Since the map $(M^*)^{\mathfrak{g}} \rightarrow (L^*)^{\mathfrak{g}}$ is onto, every element in $(L^*)^{\mathfrak{g}}$ vanishes on $x - x'$. Thus $x - x' \in \mathfrak{g}L$ and hence $x \in \mathfrak{g}M$. \square

We will also need the following lemma.

Lemma 9.5.2. *Let $C_1 \rightarrow \dots \rightarrow C_k$ be a complex of linear spaces. Suppose that each C_j is equipped with a descending filtration $F^i(C_j)$ such that $F^0(C_j) = C_j$, $\bigcap F^i(C_j) = 0$ and $\lim_{\leftarrow} C_j/F^i(C_j) \cong C_j$, and $d_j(F^i(C_j)) \subset F^i(C_{j+1})$. Suppose that $F^i(C_1)/F^{i+1}(C_1) \rightarrow \dots \rightarrow F^i(C_k)/F^{i+1}(C_k)$ is exact. Then $C_1 \rightarrow \dots \rightarrow C_k$ is also exact.*

Proof. Let $m \in \text{Ker } d_j \subset C_j$. Let $m_j := [m] \in C_j/F^i(C_j)$. We want to construct $n \in C_{j-1}$ such that $d_{j-1}(n) = m$. Since $C_{j-1} \cong \lim_{\leftarrow} C_{j-1}/F^i(C_{j-1})$ for that it is enough to construct a compatible system of representatives $n_i \in C_{j-1}/F^i(C_{j-1})$. We build this system by induction. Suppose we constructed n_i and left n'_i be an arbitrary lift of n_i to $C_{j-1}/F^{i+1}(C_{j-1})$. Consider $\varepsilon := m_{i+1} - d(n'_i)$. Then $e \in F^i(C_j)/F^{i+1}(C_j)$. Moreover, $d(\varepsilon) = 0$. Hence there exists $\delta \in F^i(C_{j-1})/F^{i+1}(C_{j-1})$ such that $d(\delta) = \varepsilon$. Define $n_{i+1} := n'_i + \delta$. \square

Proof of Lemma 7.1.9. Lemma 9.5.1 implies by induction that $\pi/F^i(\pi)$ is acyclic and $A_i := H_0(\mathfrak{g}, \pi/F^i(\pi))$ is Hausdorff, and hence is a nuclear Fréchet space. Lemma 9.5.1 also implies that $A_i \rightarrow A_{i+1}$ is onto. Define $A_{\infty} := \lim_{\leftarrow} A_i$ (as a linear, non-topological, representation). Consider the Koszul complex of π extended by A_{∞} :

$$(18) \quad 0 \leftarrow A_{\infty} \leftarrow \pi \leftarrow \mathfrak{g} \otimes \pi \leftarrow \dots \leftarrow \Lambda^n(\mathfrak{g}) \otimes \pi \leftarrow 0.$$

Lemma 9.5.2 implies that the sequence (18) is exact. Hence π is acyclic and the natural map $H_0(\mathfrak{g}, \pi) \rightarrow A_{\infty}$ is an isomorphism (of vector spaces).

Let $p_i : \pi \rightarrow \pi/F^i(\pi)$ denote the natural projection. The exactness of (18) implies

$$\mathfrak{g}\pi = \bigcap_i p_i^{-1}(\mathfrak{g}(\pi/F^i(\pi)))$$

which in turn implies that $\mathfrak{g}\pi$ is closed and thus $H_0(\mathfrak{g}, \pi)$ is a nuclear Fréchet space.

Consider the short exact sequence $0 \rightarrow F^i(\pi) \rightarrow \pi \rightarrow \pi/(F^i(\pi)) \rightarrow 0$. We showed that it consists of acyclic objects and that $H_0(\mathfrak{g}, F^i(\pi))$ and $H_0(\mathfrak{g}, \pi/(F^i(\pi)))$ are Hausdorff. By Lemma 9.5.1 this implies that $0 \rightarrow H_0(\mathfrak{g}, F^i(\pi)) \rightarrow H_0(\mathfrak{g}, \pi) \rightarrow H_0(\mathfrak{g}, \pi/(F^i(\pi))) \rightarrow 0$ is a short exact sequence of nuclear Fréchet spaces and hence the image of $H_0(\mathfrak{g}, F^i(\pi)) \hookrightarrow H_0(\mathfrak{g}, \pi)$ is closed.

Note that $F^i(H_0(\mathfrak{g}, \pi)) = H_0(\mathfrak{g}, F^i(\pi))$. To sum up, $H_0(\mathfrak{g}, \pi)$ is a nuclear Fréchet space, $F^i(H_0(\mathfrak{g}, \pi))$ are closed, the natural morphisms $F^i(H_0(\mathfrak{g}, \pi)) \rightarrow H_0(\mathfrak{g}, \pi_i)$ are isomorphisms and $H_0(\mathfrak{g}, \pi) \cong A_{\infty} \cong \lim_{\leftarrow} H_0(\mathfrak{g}, \pi)/(F^i(H_0(\mathfrak{g}, \pi)))$. \square

9.6. More on Schwartz functions on Nash manifolds. In this subsection we will recall some properties of Nash manifolds and Schwartz functions over them. We work in the notation of [AG08], where one can read about Nash manifolds and Schwartz distributions over them. More detailed references on Nash manifolds are [BCR98] and [Shi87].

Nash manifolds are equipped with the **restricted topology**, in which open sets are open semi-algebraic sets. This is not a topology in the usual sense of the word as infinite unions of open sets are not necessarily open sets in the restricted topology. However, finite unions of open sets are open and therefore in the restricted topology we consider only finite covers. In particular, if \mathcal{E} over X is a Nash vector bundle it means that there exists a finite open cover U_i of X such that $\mathcal{E}|_{U_i}$ is trivial.

Fix a Nash manifold X and a Nash bundle \mathcal{E} over X .

An important property of Nash manifolds is

Theorem 9.6.1 (Local triviality of Nash manifolds; [Shi87], Theorem I.5.12). *Any Nash manifold can be covered by a finite number of open submanifolds Nash diffeomorphic to \mathbb{R}^n .*

Together with [AG08, Corollary 3.6.3] this implies

Theorem 9.6.2. *Let $Y \subset X$ be a closed Nash submanifold. Then there exist open sunsets $\{U_i\}_{i=1}^n \subset X$ and Nash diffeomorphisms $\phi_i : \mathbb{R}^n \xrightarrow{\sim} U_i$ such that $Y \subset \bigcup_{i=1}^n U_i$ and $\phi_i^{-1}(U_i \cap Y)$ is a linear subspace of \mathbb{R}^n .*

Theorem 9.6.3 ([AG10], Theorem 2.4.16). *Let $s : X \rightarrow Y$ be a surjective submersive Nash map. Then locally it has a Nash section, i.e. there exists a finite open cover $Y = \bigcup_{i=1}^k U_i$ such that s has a Nash section on each U_i .*

Corollary 9.6.4. *An étale map $\phi : X \rightarrow Y$ of Nash manifolds is locally an isomorphism. That means that there exists a finite cover $X = \bigcup U_i$ such that $\phi|_{U_i}$ is an isomorphism onto its open image.*

Corollary 9.6.5 ([AG09], Theorem B.2.3). *Let $p : X \rightarrow Y$ be a Nash submersion of Nash manifolds. Then there exist a finite open (semi-algebraic) cover $X = \bigcup U_i$ and isomorphisms $\phi_i : U_i \cong W_i$ and $\psi_i : p(U_i) \cong \mathcal{O}_i$ where $W_i \subset \mathbb{R}^{d_i}$ and $\mathcal{O}_i \subset \mathbb{R}^{k_i}$ are open (semi-algebraic) subsets, $k_i \leq d_i$ and $p|_{U_i}$ correspond to the standard projections.*

Notation 9.6.6. *Let $U \subset X$ be an open (Nash) subset. We denote by $\text{Nash}(U, \mathcal{E})$ the space of Nash sections of \mathcal{E} on U .*

Definition 9.6.7. *We call a Nash action of a Nash group G on X **strictly simple** if it is simple (i.e. all stabilizers are trivial) and there exists a geometric (separated) Nash quotient $G \setminus X$ (see [AG10, Definition 4.0.4]).*

Proposition 9.6.8 ([AG10], Proposition 4.0.14). *Let $H < G < G_n$ be Nash groups. Then the action of H on G is strictly simple.*

Corollary 9.6.9. *Let $H < G < G_n$ be Nash groups. Let X be a transitive Nash G -manifold. Then there exists a geometric quotient $H \setminus X$.*

We will use the following theorem that describes the basic properties of Schwartz functions on Nash manifolds.

Theorem 9.6.10.

- (i) $\mathcal{S}(\mathbb{R}^n) = \text{Classical Schwartz functions on } \mathbb{R}^n$.
- (ii) The space $\mathcal{S}(X, \mathcal{E})$ is a nuclear Fréchet space.
- (iii) Let $Z \subset X$ be a closed Nash submanifold. Then the restriction maps $\mathcal{S}(X, \mathcal{E})$ onto $\mathcal{S}(Z, \mathcal{E}|_Z)$.
- (iv) Let $X = \bigcup U_i$ be a finite open cover of X . Then a global section f of \mathcal{E} on X is a Schwartz section if and only if it can be written as $f = \sum_{i=1}^n f_i$ where $f_i \in \mathcal{S}(U_i, \mathcal{E})$ (extended by zero to X). Moreover, there exists a tempered partition of unity $1 = \sum_{i=1}^n \lambda_i$ such that for any Schwartz section $f \in \mathcal{S}(X, \mathcal{E})$ the section $\lambda_i f$ is a Schwartz section of \mathcal{E} on U_i (extended by zero to X). Note that this property implies that Schwartz sections form a cosheaf in the restricted topology.
- (v) Tempered sections of a Nash bundle \mathcal{E} over a Nash manifold X form a sheaf in the restricted topology. Namely, let U_i be a (finite) open Nash cover of X and let $s \in \Gamma(X, \mathcal{E})$. Then s is a tempered section if and only if $s|_{U_i}$ is a tempered section for any i .
- (vi) For any Nash manifold Y ,

$$\mathcal{S}(X \times Y) = \mathcal{S}(X) \hat{\otimes} \mathcal{S}(Y).$$

Part (i) is [AG08, Theorem 4.1.3], part (ii) is [AG10, Corollary 2.6.2], for (iii) see [AG08, §§1.5], for (iv) and (v) see [AG08, §§5], for (vi) see e.g. [AG10, Corollary 2.6.3].

Lemma 9.6.11 ([AOS], Appendix A). *Suppose*

$$0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_2 \rightarrow \mathcal{E}_3 \rightarrow 0$$

is an exact sequence of Nash bundles on X . Then

$$0 \rightarrow \mathcal{S}(X, \mathcal{E}_1) \rightarrow \mathcal{S}(X, \mathcal{E}_2) \rightarrow \mathcal{S}(X, \mathcal{E}_3) \rightarrow 0.$$

is an exact sequence of Fréchet spaces.

For the proof we will need the following lemma.

Lemma 9.6.12. *Suppose*

$$0 \rightarrow \mathcal{E}_1 \xrightarrow{d_1} \mathcal{E}_2 \xrightarrow{d_2} \mathcal{E}_3 \rightarrow 0$$

is a finite exact sequence of Nash bundles on X . Then there exist an open Nash cover $X = \bigcup_1^n U_j$ s.t. the sequence

$$0 \rightarrow \mathcal{E}_1|_{U_j} \xrightarrow{d_1} \mathcal{E}_2|_{U_j} \xrightarrow{d_2} \mathcal{E}_3|_{U_j} \rightarrow 0$$

is isomorphic to the sequence

$$(19) \quad 0 \rightarrow U_j \times V_1 \rightarrow U_j \times (V_1 \oplus V_2) \rightarrow U_j \times V_2 \rightarrow 0$$

Where V_i are finite dimensional vector spaces, $U_i \times V$ denotes the constant bundle with fiber V and the maps in the sequence come from the the standard embedding and projection.

This lemma is essential proven in [BCR98], but for completeness let us prove it here.

Proof. By the definition of a Nash bundle we can find a finite Nash open cover $X = \bigcup_1^n V_j$ s.t. the bundles $\mathcal{E}_i|_{V_i}$ are constant. Thus we may assume that \mathcal{E}_i are constant bundles with fibers W_i . Choose a basis for W_2 and let I be the collection of coordinate subspaces. For any $V \in I$ denote $U_V = \{x \in X | ((d_2)_x)|_V$ is an isomorphism}. Clearly U_V form a Nash cover of X . Thus we may assume that $X = U_V$ for some V . in this case d_1 gives an isomorphism between the constant bundle $X \times (W_1 \oplus V)$ and the constant bundle $X \times W_2$. Also d_2 gives an isomorphism between $X \times V$ and $X \times W_3$. Those isomorphisms gives identification of the sequence

$$0 \rightarrow X \times W_1 \rightarrow X \times W_2 \rightarrow X \times W_3 \rightarrow 0$$

with

$$0 \rightarrow X \times W_1 \rightarrow X \times (W_1 \oplus V) \rightarrow X \times V \rightarrow 0.$$

□

Proof of Lemma 9.6.11.

Step 1 The case when the sequence is as in (19).

It follows immediately from the definition of Schwartz section of a Nash bundle.

Step 2 The general case

Let $X = \bigcup U_j$ be a finite open Nash cover s.t. the sequence

$$0 \rightarrow \mathcal{E}_1|_{U_j} \xrightarrow{d_1} \mathcal{E}_2|_{U_j} \xrightarrow{d_2} \mathcal{E}_3|_{U_j} \rightarrow 0$$

is isomorphic to a sequence

$$0 \rightarrow U_j \times V_1 \rightarrow U_j \times (V_1 \oplus V_2) \rightarrow U_j \times V_2 \rightarrow 0$$

as in Lemma 9.6.12. By Theorem 9.6.10(iv) we can choose a partition of unity $1_X = \sum e_j$ where $e_j \in C^\infty(X)$ with $\text{Supp}(e_j) \subset U_j$ and such that for any i, j and any $\phi \in \mathcal{S}(X, \mathcal{E}_i)$ we have $e_j \phi \in \mathcal{S}(U_j, E_i)$. Let $\phi \in \text{Ker}(d_i) \subset \mathcal{S}(X, \mathcal{E}_i)$ then we have

$$\phi = \sum e_j \phi \in \sum \text{Ker}(d_i|_{\mathcal{S}(U_j, \mathcal{E}_i|_{U_j})}) = \sum \text{Im}(d_{i-1}|_{\mathcal{S}(U_j, \mathcal{E}_{i-1}|_{U_j})}) \subset \text{Im}(d_{i-1}).$$

□

The following Lemma follows immediately from Corollary 9.6.5 and Theorem 9.6.10(v).

Lemma 9.6.13. *Let $p : Y \rightarrow X$ be a Nash surjective submersion of Nash manifolds. Let $s \in \Gamma(X, \mathcal{E})$ be a set-theoretical global section of \mathcal{E} . Then s is tempered if and only if p^*s is a tempered section of $p^*\mathcal{E}$.*

Lemma 9.6.14 ([AG09], Theorem B.2.4). *Let $\phi : Y \rightarrow X$ be a Nash submersion of Nash manifolds. Then*

(i) there exists a unique continuous linear map $\phi_ : \mathcal{S}(Y, \phi^?(\mathcal{E})) \rightarrow \mathcal{S}(X, \mathcal{E})$ such that for any $f \in \mathcal{S}(X, \mathcal{E}^* \otimes D_X)$ and $\mu \in \mathcal{S}(Y, \phi^?(\mathcal{E}))$ we have*

$$\int_{x \in X} \langle f(x), \phi_* \mu(x) \rangle = \int_{y \in Y} \langle \phi^* f(y), \mu(y) \rangle.$$

In particular, we mean that both integrals converge. Here, $\phi^?(\mathcal{E}) = \phi^(\mathcal{E}) \otimes D_Y^X$, as in Notation 6.0.5.*

(ii) If ϕ is surjective then ϕ_ is surjective.*

Lemma 9.6.15. *[AG] , Lemma B.1.4] Let G be a connected Nash group and \mathfrak{g} be the Lie algebra of G . Let $p : G \times X \rightarrow X$ be the projection. Let G act on $\mathcal{S}(G \times X, p^?(\mathcal{E}))$ by acting on the G coordinate. Consider the pushforward $p_* : \mathcal{S}(G \times X, p^?(\mathcal{E})) \rightarrow \mathcal{S}(X, \mathcal{E})$. Then $\mathfrak{g}\mathcal{S}(G \times X, p^?(\mathcal{E})) = \text{Ker}(p_*)$.*

Lemma 9.6.16. *Let $f : X \rightarrow \mathbb{R}$ be a Nash function such that 0 is a regular value of f . Then $f\mathcal{S}(X) = \{\phi \in \mathcal{S}(X) : \phi(f^{-1}(0)) = 0\}$.*

Proof. By Theorem 9.6.2 and partition of unity (Theorem 9.6.10(iv)), we can assume that $X = \mathbb{R}^n$ and $f^{-1}(0) = \mathbb{R}^{n-1} \subset \mathbb{R}^n$. Let $x : \mathbb{R}^n \rightarrow \mathbb{R}$ be the last coordinate function. Since 0 is a regular value of f , f/x is a smooth invertible function. Since f is a Nash function, f/x is also a Nash function. Thus, we can assume $f = x$. We have to show that the following sequence is exact.

$$0 \rightarrow \mathcal{S}(\mathbb{R}^n) \xrightarrow{x} \mathcal{S}(\mathbb{R}^n) \xrightarrow{\text{Res}} \mathcal{S}(\mathbb{R}^{n-1}) \rightarrow 0,$$

where $\text{Res} : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^{n-1})$ is the restriction map.

By the Schwartz Kernel Theorem (Proposition vi) this sequence is isomorphic to

$$0 \rightarrow \mathcal{S}(\mathbb{R}^{n-1}) \widehat{\otimes} \mathcal{S}(\mathbb{R}) \xrightarrow{\text{Id} \widehat{\otimes} x} \mathcal{S}(\mathbb{R}^{n-1}) \widehat{\otimes} \mathcal{S}(\mathbb{R}) \xrightarrow{\text{Id} \widehat{\otimes} \text{Res}} \mathcal{S}(\mathbb{R}^{n-1}) \rightarrow 0,$$

where $\text{Res} : \mathcal{S}(\mathbb{R}) \rightarrow \mathbb{C}$ is evaluation at 0.

By Proposition 9.4.4 it is enough to prove the exactness in the case $n = 1$, which is obvious. \square

We will use the following version of Borel's lemma.

Lemma 9.6.17. *Let $Z \subset X$ be Nash submanifold.*

Then $\mathcal{S}_X(Z)$ has a canonical countable decreasing filtration by closed subspaces $(\mathcal{S}_X(Z))^i$ satisfying

- (1) $\bigcap (\mathcal{S}_X(Z))^i = 0$
- (2) $\text{gr}_i(\mathcal{S}_X(Z, \mathcal{E})) \cong \mathcal{S}(Z, \text{Sym}^i(CN_Z^X) \otimes \mathcal{E})$
- (3) the natural map

$$\mathcal{S}_X(Z, \mathcal{E}) \rightarrow \varprojlim (\mathcal{S}_X(Z, \mathcal{E}) / \mathcal{S}_X(Z, \mathcal{E}))^i$$

is an isomorphism.

For proof see [AG, Lemmas B.0.8 and B.0.9].

9.7. Good representations of geometric origin (Proof of Lemma 7.1.6). For the proof we will fix T, R and R' . We will need the following statements.

Lemma 9.7.1. *Let X be an R -transitive Nash manifold. Let \mathcal{E} be an R -tempered bundle on X . Suppose that for each $x \in X$, the V_n -stabilizer of x acts trivially on the fiber \mathcal{E}_x . Note that by Corollary 9.6.9 there exists geometric quotient $V_n \setminus X$. Let $p : X \rightarrow V_n \setminus X$ denote the projection. Then*

- (1) *there exists an R' -tempered bundle \mathcal{E}' on the geometric quotient $V_n \setminus X$ and a (tempered) isomorphism of R -tempered bundles $\mathcal{E} \xrightarrow{\sim} p^*(\mathcal{E}')$.*
- (2) *there exists an R' -tempered bundle \mathcal{E}'' on the geometric quotient $V_n \setminus X$ and a (tempered) isomorphism of R -tempered bundles $\mathcal{E} \xrightarrow{\sim} p^?(\mathcal{E}'')$.*

Proof. (1) Let $x \in X$ and let R_x be the stabilizer of x . Identify $V_n \setminus X$ with $Y := R'x$. Let $\mathcal{E}' := \mathcal{E}|_Y$.

We will construct a (tempered) isomorphism of R -tempered bundles $\phi : \mathcal{E} \xrightarrow{\sim} p^*(\mathcal{E}')$. In order to do that we have to construct, for any $x \in X$, an element $\phi_x \in \text{Hom}(\mathcal{E}_x, p^*(\mathcal{E}')_x)$. Note that $\text{Hom}(\mathcal{E}_x, p^*(\mathcal{E}'))_x = \text{Hom}(\mathcal{E}_x, \mathcal{E}_{p(x)})$. We know that there exists $g \in V_n$ such that $gx = p(x)$. This g gives an element in $\phi_x \in \text{Hom}(\mathcal{E}_x, p^*(\mathcal{E}'))_x$. The element g is not unique, but the assumptions of the lemma imply that ϕ_x does not depend on the choice of g . Now we have to show that the constructed $\phi \in \Gamma(X, \text{Hom}(\mathcal{E}, p^*(\mathcal{E}')))$ is a tempered section. Consider the action map $a : V_n \times Y \rightarrow X$. Clearly, it is a surjective submersion. It is easy to see that $a^*(\phi) \in \Gamma(V_n \times Y, \text{Hom}(a^*\mathcal{E}, a^*p^*(\mathcal{E}')))$ is a tempered section. Thus, Lemma 9.6.13 implies that $\phi \in \Gamma(X, \text{Hom}(\mathcal{E}, p^*(\mathcal{E}')))$ is a tempered section.

(2) By part (1) there exists a bundle $(D_Y^X)'$ on $V_n \setminus X$ such that $p^*((D_Y^X)') \simeq D_Y^X$. We define $\mathcal{E}'' := \mathcal{E}' \otimes ((D_Y^X)')^*$. \square

Remark 9.7.2. An analog of this lemma holds for Nash equivariant bundles, and for multiplicative equivariant bundles, with an analogous proof. In particular, the isomorphism $p^*((D_Y^X)') \simeq D_Y^X$ is an isomorphism of Nash bundles.

Corollary 9.7.3. Let X be a transitive R -Nash manifold. Let \mathcal{E} be a R -multiplicative bundle on X (see Definition 6.1.4). Then there exists a filtration F^i of \mathcal{E} by R -multiplicative bundles such that for every i there exists an R' -multiplicative bundle \mathcal{E}'_i on the geometric quotient $V_n \setminus X$ such that $F^i(\mathcal{E})/F^{i-1}(\mathcal{E}) = p^?(\mathcal{E}'_i)$.

To deduce this corollary we will use the following obvious lemma.

Lemma 9.7.4. Let π be a multiplicative (finite dimensional) representation of a subgroup L of V_n . Then the Lie algebra of L acts nilpotently on π .

Proof of Corollary 9.7.3. Let $F^i(\mathcal{E}) \subset \mathcal{E}$ be defined by $F^i(\mathcal{E}) := \{(x, e) \in \mathcal{E} | ((\mathfrak{v}_n)_x)^{\otimes i} e = 0\}$. By the last lemma (Lemma 9.7.4) the filtration $F^i(\mathcal{E})$ is exhaustive. Since \mathcal{E} is a tempered bundle, the action of the Lie algebra \mathfrak{r} is Nash and hence $F^i(\mathcal{E})$ are Nash. Clearly the action $(\mathfrak{v}_n)_x$ on $F^i(\mathcal{E})/F^{i-1}(\mathcal{E})$ is trivial. Thus, by Lemma 9.7.1, we are done. \square

Proof of Lemma 7.1.6. The case $Y = X$ and X is R -transitive follows from Corollary 9.7.3 and Lemma 9.6.11. By the Borel lemma (Lemma 9.6.17), this implies the case when X is R -transitive and Y is arbitrary. This in turn implies the general case by Corollary 6.0.4. \square

9.8. Induced representations as sections of multiplicative bundles (Proof of Lemma 7.1.7). Identify the variety $X := G/H$ with $K/(K \cap H)$. Note that under this identification $\mathcal{E} \cong K \times_{K \cap H} \chi$. This is a Nash bundle.

Let us show that the action of G on \mathcal{E} is multiplicative. The only non-trivial part is that any $\alpha \in \mathfrak{g}_n$ preserve the space of Nash sections of \mathcal{E} on any open (Nash) set. For this we note that

$$\text{Nash}(U, \mathcal{E}) \cong \{f \in C^\infty(p^{-1}(U)) | f|_K \text{ is Nash and } f(gh) = \chi(h)f(g) \text{ for any } h \in H\},$$

where $p : G \rightarrow X$ is the quotient map. Under this identification, an element $\alpha \in \mathfrak{g}$ acts on $\text{Nash}(U, \mathcal{E})$ via a Nash vector field β on $p^{-1}(U)$. This field can be interpreted as a Nash map $\beta : p^{-1}(U) \rightarrow \mathfrak{g}$. Here we identify the Lie algebra \mathfrak{g} with the tangent space at a point $g \in G$ using the differential of the left translation by g . Restrict β to $K \cap p^{-1}(U)$ and decompose it to a sum of $\beta_1 : K \cap p^{-1}(U) \rightarrow \mathfrak{k}$ and $\beta_2 : K \cap p^{-1}(U) \rightarrow \mathfrak{h}$, where \mathfrak{h} and \mathfrak{k} denote the Lie algebras of H and K respectively. Let $f \in C^\infty(p^{-1}(U))$ such that $f|_K$ is Nash and $f(gh) = \chi(h)f(g)$ for any $h \in H$. Then $(\beta f)|_K = \beta_1 f|_K + d\chi \circ \beta_2 \cdot f$ which is clearly a Nash function. \square

9.9. Derivatives of quasi-regular representations on P_n -orbits on flag varieties (Proofs of Lemmas 8.4.1 and 8.4.3).

Proof of Lemma 8.4.1. The proof is by induction on n . In step 1 we reduce to the case when $Y = X = P_n/Q_\lambda^i$. Let Z and Z_0 to be as in Lemma 8.3.6 and $p : X \rightarrow Z$ denote the natural projection. In step 2 we reduce to the case $\mathcal{E} = p^?(\mathcal{E}')$, where \mathcal{E}' is a G_{n-1} -multiplicative bundle on Z . In step 3 we prove the lemma for this case.

Step 1 Reduction to the case $Y = X = P_n/Q_\lambda^i$

By Lemma 9.6.17, $\mathcal{S}_Y(X, \mathcal{E})$ is a good extension of $\pi_i := \mathcal{S}(X, \mathcal{E}|_X \otimes \text{Sym}^i(CN_X^Y))$. By Lemma 7.1.6, π_i are good. Thus, by Corollary 7.1.10, $E^{i+1}(\pi)$ is a good extension of $E^{i+1}(\pi_i)$. Thus it is enough to show that $E^{i+1}(\pi_i) = 0$.

Step 2 Reduction to the case $\mathcal{E} = p^?(\mathcal{E}')$, where \mathcal{E}' is a G_{n-1} -multiplicative bundle on Z .

By Corollary 9.7.3, we have a finite filtration on \mathcal{E} and G_{n-1} -multiplicative bundles \mathcal{E}'_i on Z such that $p^?(\mathcal{E}_i) \simeq \text{Gr}^i(\mathcal{E})$. By Lemma 9.6.11, this gives a filtration on $\mathcal{S}(X, \mathcal{E})$ such that $\text{Gr}^i(\mathcal{S}(X, \mathcal{E})) = \mathcal{S}(X, p^?(\mathcal{E}_i))$. As before, this means that it is enough to show that $E^{i+1}(\mathcal{S}(X, p^?(\mathcal{E}_i))) = 0$.

Step 3 Proof for the case $\mathcal{E} = p^?(\mathcal{E}')$.

By the analytic key lemma (Lemma 7.1.1), $\tilde{\Phi}(\pi) = \mathcal{S}(Z_0, \mathcal{E}')$. Denote $O_j := P_{n-1}x_\lambda^{ij}$ for $1 \leq j < i$. By Corollary 8.3.4 and Lemma 8.3.6, $Z_0 = \bigcup_{j=1}^{i-1} \mathcal{O}_\lambda^{ij}$. By Corollary 6.0.4, there is a filtration F^j on $\mathcal{S}(Z_0, \mathcal{E}')$ such that $\text{Gr}^j(\mathcal{S}(Z_0, \mathcal{E}')) = \mathcal{S}_{Z_0}(\mathcal{O}_\lambda^{ij}, \mathcal{E}')$. Thus it is enough to show that $E^i(\mathcal{S}_{Z_0}(\mathcal{O}_\lambda^{ij}, \mathcal{E}')) = 0$ for any j . By Corollary 8.3.4, $\mathcal{O}_\lambda^{ij} \cong P_{n-1}/Q_{\lambda'}^j$. Thus, by the induction hypothesis, $E^i(\mathcal{S}_{Z_0}(\mathcal{O}_\lambda^{ij}, \mathcal{E}')) = 0$.

□

For the proof of Lemma 8.4.3, consider the embedding of G_{n-1} to G_n obtained by conjugating the standard embedding by the permutation matrix corresponding to the permutation $(m_\lambda^i, m_\lambda^i + 1, \dots, n)$. Under this embedding, $P_{\lambda_i^-}$ embeds into P_λ . Denote $\beta_k^i := (|\det|^{-1/2}, \dots, |\det|^{-1/2}, 1, |\det|^{1/2}, \dots, |\det|^{1/2}) \in \mathfrak{X}^k$, where 1 stands on place number i . We will need the following straightforward computation.

Lemma 9.9.1. *Let $\alpha \in \mathfrak{X}^k$. Then*

$$\chi_{\lambda, \alpha}|_{P_{\lambda_i^-}} = \chi_{\lambda_i^-, \alpha \cdot \beta_k^i},$$

where $\alpha \cdot \beta_k^i \in \mathfrak{X}^k$ denotes the coordinate-wise product.

Proof of Lemma 8.4.3. The proof is by induction. Let us first prove the base $i = 1$. Note that $X = P_n/Q_\lambda^1 \cong G_{n-1}/P_{\lambda_k^-}$. Thus $\pi|_{G_{n-1}} = \mathcal{S}(G_{n-1}/P_{\lambda_k^-}, \mathcal{E})$. Consider the fiber of \mathcal{E} at the class of 1 as a character of $P_{\lambda_k^-}$. We get $\mathcal{E}_{x_0} = \chi_{\lambda, \alpha}|_{P_{\lambda_k^-}} = \chi_{\lambda_k^-, \alpha \cdot \beta_k^1}$. Thus

$$\pi|_{G_{n-1}} = \chi_{(n_1), (\alpha_1 |\det|^{-1/2})} \times \cdots \times \chi_{(n_{k-1}), (\alpha_{k-1} |\det|^{-1/2})} \times \chi_{(n_k-1), (\alpha_k)}.$$

For the induction step, we assume $i > 0$ and let Z , Z_0 and λ' be as in Lemma 8.3.6 and $p : X \rightarrow Z$ denote the natural projection. By Lemma 9.7.1 (and Remark 9.7.2) there exists a G_{n-1} -multiplicative equivariant bundle \mathcal{E}' on Z such that $\mathcal{E} = p^?(\mathcal{E}')$. By the analytic key lemma (Lemma 7.1.1), $\Phi(\pi) = \mathcal{S}(Z_0, \mathcal{E}') \otimes |\det|^{-1/2}$. By Corollary 8.3.4 and Lemma 8.3.6, $Z_0 = \bigcup_{j=1}^{i-1} \mathcal{O}_{\lambda'}^{ij}$. By Corollary 6.0.4, there is a filtration F^j on $\mathcal{S}(Z_0, \mathcal{E}')$ such that $\text{Gr}^j(\mathcal{S}(Z_0, \mathcal{E}')) = \mathcal{S}_{Z_0}(\mathcal{O}_{\lambda'}^{ij}, \mathcal{E}')$. By Corollary 8.3.4, $\mathcal{O}_{\lambda'}^{ij} \cong P_{n-1}/Q_{\lambda'}^j$. Thus, by Lemma 8.4.1, $\Phi^{i-2}(\mathcal{S}_{Z_0}(\mathcal{O}_{\lambda'}^{ij}, \mathcal{E}')) = 0$ for $j < i-1$. Thus

$$\begin{aligned} E^i(\pi) &= \Phi^{i-1}(\pi) = \Phi^{i-2}(\Phi(\pi)) = \Phi^{i-2}(\mathcal{S}_{Z_0}(\mathcal{O}_{\lambda'}^{i, i-1}, \mathcal{E}') \otimes |\det|^{-1/2}) = \\ &= \Phi^{i-2}(\mathcal{S}(\mathcal{O}_{\lambda'}^{i, i-1}, \mathcal{E}' \otimes |\det|^{-1/2})) = E^{i-1}(\mathcal{S}(\mathcal{O}_{\lambda'}^{i, i-1}, \mathcal{E}' \otimes |\det|^{-1/2})). \end{aligned}$$

In the last expression we consider the character $|\det|^{-1/2}$ of P_{n-1} as a constant P_{n-1} -equivariant bundle on Z_0 .

Consider $\nu_1 := \mathcal{E}'|_{x_\lambda^{i, i-1}}$ and $\nu_2 := \mathcal{E}|_{x_\lambda^{i, i-1}}$ as characters of $(P_{n-1})_{x_\lambda^{i, i-1}} = Q_{\lambda'}^{i-1}$. Note that

$$\mathcal{E}'|_{x_{\lambda'}^{i, i-1}} = \mathcal{E}|_{x_{\lambda'}^{i, i-1}} \otimes D_X^*|_{x_{\lambda'}^{i, i-1}} \otimes D_Z|_{x_{\lambda'}^{i, i-1}}$$

and thus $\nu_1 = \nu_2 \otimes \overline{\chi}_{\lambda', \beta}^{i-1}|_{Q_{\lambda'}^{i-1}}$, where $\beta = (|\det|, \dots, |\det|, 1, \dots, 1)$, where the last appearance of $|\det|$ is in the place $k-i+1$. Thus

$$\begin{aligned} (\mathcal{E}' \otimes |\det|^{-1/2})|_{x_{\lambda}^{i,i-1}} &= \nu_1 \otimes |\det|^{-1/2} = \nu_2 \cdot \overline{\chi}_{\lambda', \beta}^{i-1}|_{Q_{\lambda'}^{i-1}} \cdot |\det|^{-1/2} = \\ (\chi_{\lambda, \alpha}^i|_{Q_{\lambda'}^{i-1}} \cdot \overline{\chi}_{\lambda', \beta}^{i-1}|_{Q_{\lambda'}^{i-1}} \cdot |\det|^{-1/2})| &= ((\chi_{\lambda, \alpha}^i|_{P_{\lambda'}^{i-1}}) \cdot \overline{\chi}_{\lambda', \beta}^{i-1} \cdot |\det|^{-1/2})|_{Q_{\lambda'}^{i-1}} = \\ (\chi_{\lambda', \alpha \cdot \beta_k^{k-i+1}}^{i-1} \cdot \overline{\chi}_{\lambda', \beta}^{i-1} \cdot |\det|^{-1/2})|_{Q_{\lambda'}^{i-1}} &= \chi_{\lambda', \alpha'}^{i-1}|_{Q_{\lambda'}^{i-1}}, \end{aligned}$$

where $\alpha' = (\alpha_1, \dots, \alpha_{k-i}, \alpha_{k-i+1}|\det|^{1/2}, \alpha_{k-i+2}, \dots, \alpha_k)$.

Thus, by the induction hypothesis,

$$\begin{aligned} E^i(\pi)|_{G_{n-i}} &= E^{i-1}(\mathcal{S}(\mathcal{O}_{\lambda}^{i-1, j}, \mathcal{E}' \otimes |\det|^{-1/2})) = \\ \chi_{(n_1), (\alpha_1|\det|^{-1/2})} \times \cdots \times \chi_{(n_{k-i}), (\alpha_{k-i}|\det|^{-1/2})} \times \chi_{(n_{k-i+1}-1), (\alpha_{k-i+1})} \times \cdots \times \chi_{(n_k-1), (\alpha_k)} \end{aligned}$$

□

10. HOMOLOGY OF GEOMETRIC REPRESENTATIONS AND THE PROOF OF THE ANALYTIC KEY LEMMA (LEMMA 7.1.1)

10.1. Sketch of the proof. We have to compute \mathfrak{v}_n -homology of the representation $\mathcal{S}(X, \mathcal{E}) \otimes (-\psi)$. We first note that this task is local on $X' = V_n \setminus X$, i.e. if we cover X' by subsets U_i it is enough to compute the homology of $\mathcal{S}(p^{-1}(U_i), \mathcal{E}) \otimes (-\psi)$. Now we can cover X' by refined enough cover s.t. for each U_i , the space $p^{-1}(U_i)$ will look like a product $U_i \times W$ where W is a V_n -orbit and the bundle $\mathcal{E}|_{U_i}$ is trivial. Thus we reduced to computation of homology of $\mathcal{S}(U_i \times W) \otimes (-\psi)$.

Note that the action of V_n on $U_i \times W$ is not the usual product action but rather a twisted product, i.e. the action V_n on its orbit $\{x\} \times W$ depends on the point $x \in U_i$. We can untwist this product (see Proposition 10.2.5) but this will cause a twist in the character ψ . Namely, it will replace it by a line bundle \mathcal{E} where the action of V_n on $\mathcal{E}_{\{x\} \times W}$ depends on the point $x \in U_i$. Thus we reduce to computation of homology of $\mathcal{S}(U_i \times W, \mathcal{E})$.

Now we use a relative version of the Shapiro Lemma (see Theorem 10.2.1) in order to reduce to the computation of the homology of $\mathcal{S}(U_i, \mathcal{E})$ as a representation of the stabilizer $(V_n)_0$ in V_n of a point $0 \in W$. Note that the action of $(V_n)_0$ on U_i is trivial and thus we can view $\mathcal{S}(U_i, \mathcal{E})$ as a family of characters, i.e. it is defined by a map ϕ from U_i to the space $(V_n)_0^*$ of characters of $(V_n)_0$. In Lemma 10.2.2 we compute homology of such families under the assumption that ϕ is submersive at the trivial character. This assumption is satisfied in our case due to the action of P_n .

Remark 10.1.1. *The proof is based on series of reductions. If we were interested only in acyclicity then one could give a relatively simple proof in which each of those reductions is given by general statements. However we are also interested in computation of H_0 and this makes those reductions more complicated. For example in the first step when we say that the computation is local one should explain what does this mean. We do it by constructing an explicit morphism $I : H_0(\mathfrak{v}_n, \mathcal{S}(X, \mathcal{E}) \otimes (-\psi)) \rightarrow \mathcal{S}(X'_0, \mathcal{E}|_{X'_0})$ and proving that this morphism is an isomorphism rather than proving that there exists some isomorphism. This forces us to make each reduction more explicit. This is sometimes unpleasant since some of the reductions e.g. the Shapiro lemma are using the general machinery of homological algebra which usually is not so explicit.*

Therefore if the reader is not interested in all the details we recommend him to skip all the parts that regard H_0 and concentrate only on the acyclicity. The computation of H_0 is essentially the same but its exposition is much longer.

10.2. Ingredients of the proof. We will need the following version of Shapiro lemma.

Theorem 10.2.1 (Relative Shapiro lemma). *Let G be an affine Nash group and X be a transitive Nash G -manifold. Let Y be a Nash manifold. Let $x \in X$ and denote $H := G_x$. Let $\mathcal{E} \rightarrow X \times Y$ be a G equivariant Nash bundle. Suppose that G and H are homologically trivial (i.e. all their homology except H_0 vanish and $H_0 = \mathbb{R}$). Let $\mathfrak{X} : G \times Y \rightarrow \mathbb{R}$ be a Nash map such that for any $y \in Y$, the map $\mathfrak{X}|_{G \times \{y\}}$ is a group homomorphism. Let \mathfrak{X}' be 1-dimensional G -equivariant bundle on Y , with action of G given by*

$g(y, v) = (y, \theta(\mathfrak{X}(g, y))v)$. Let $\mathcal{E}' := \mathcal{E} \otimes (\mathbb{C} \boxtimes \mathfrak{X}')$, where \mathbb{C} denotes the trivial bundle on X and \boxtimes denotes exterior tensor product. Then

$$H_i(\mathfrak{g}, \mathcal{S}(X \times Y, \mathcal{E}')) \cong H_i(\mathfrak{h}, \mathcal{S}(\{x\} \times Y, \mathcal{E}'_{\{x\} \times Y} \otimes \Delta_H \cdot (\Delta_G^{-1})|_H)),$$

where Δ_H and Δ_G denote the modular characters of the groups H and G .

The proof of this theorem is along the lines of the proof of [AG10, Theorem 4.0.9]. We give it in Appendix A.

Lemma 10.2.2. *Let X be a Nash manifold and V be a real vector space. Let $\phi : X \rightarrow V^*$ be a Nash map. Suppose that $0 \in V^*$ is a regular value of ϕ . It gives a map $\chi : V \rightarrow \mathcal{T}(X)$ given by $\chi(v)(x) = \theta(\phi(x)(v))$ (recall that $\mathcal{T}(X)$ denotes the space of tempered functions). This gives an action of V on $\mathcal{S}(X)$ by $\pi(v)(f) := \chi(v) \cdot f$. Then*

- (i) $H_i(\mathfrak{v}, \mathcal{S}(X)) = 0$ for $i > 0$.
- (ii) Let $X_0 := \phi^{-1}(0)$. Note that it is smooth. Let r denote the restriction map $r : \mathcal{S}(X) \rightarrow \mathcal{S}(X_0)$. Then r gives an isomorphism $H_0(\mathfrak{v}, \mathcal{S}(X)) \xrightarrow{\sim} \mathcal{S}(X_0)$.

We will prove this lemma in §§10.5

Notation 10.2.3. *In the situation of the Lemma we will denote the action of \mathfrak{v} on $\mathcal{S}(X)$ by $\tau_{X, \phi}$.*

In order to check the conditions of Lemma 10.2.2 we will need the following lemma that we will prove in §§10.6.

Lemma 10.2.4. *Let a Nash group G act linearly on a finite-dimensional vector space V over F , such that the action on $V^* \setminus 0$ is transitive. Let Q be a closed Nash subgroup of G and L be a vector subspace of V stabilized by Q . Let $\varphi \in V^*$ be a functional. Consider the map $a : G \rightarrow L^*$ defined by $a(g) := g\varphi|_L$. Let $U \subset G/Q$ be an open (Nash) subset and $s : U \rightarrow G$ be a local Nash section of the canonical projection $p : G \rightarrow G/Q$. Then 0 is a regular value of $\mu := a \circ s$.*

Proposition 10.2.5. *Let a Nash group G act transitively on a Nash manifold Y and let X be a Nash manifold.*

Let X “act” on G i.e. let G' be a Nash group acting on G by automorphisms and $a : X \rightarrow G'$ be a Nash map. This defines a twisted action of G on $X \times Y$. More precisely $\rho_1(g)(x, y) = (x, a(x)(g)(y))$. Let χ be a fixed tempered character of G . Let ρ_2 denote the non-twisted action of G on $X \times Y$, i.e. $\rho_2(g)(x, y) = (gx, y)$.

Define function $\mathfrak{X}(g, x, y) = \chi((a(x))^{-1}(g))$. Note that it does not depend on y . It defines a tempered (G, ρ_2) -equivariant structure on the trivial line bundle on $X \times Y$. We denote the resulting bundle by \mathcal{E} . Let π_1 denote the representation of \mathfrak{g} on $\mathcal{S}(X \times Y) \otimes \chi$ given by the action ρ_1 and π_2 denote the representation of \mathfrak{g} on $\mathcal{S}(X \times Y, \mathcal{E})$ given by the action ρ_2 .

Then $H_*(\mathfrak{g}, \pi_1) = H_*(\mathfrak{g}, \pi_2)$.

We will prove this proposition in §§10.4.

Notation 10.2.6. *In the situation of the proposition we will denote ρ_1 by $\rho_{a, X, Y}$ and π_1 by $\pi_{a, X, Y}$.*

10.3. Proof of the analytic key lemma. First, let us prove the following version of the lemma.

Lemma 10.3.1. *Let X' be a Nash manifold. Let $0 \rightarrow L \rightarrow V \xrightarrow{p} W \rightarrow 0$ be an exact sequence of finite dimensional vector spaces over F . Let $\nu : X' \rightarrow V^*$ be a Nash map, such that 0 is a regular value of the composition $X' \rightarrow V^* \rightarrow L^*$. Let X'_0 be the preimage of 0 under this composition. Note that it is smooth. Fix a Haar measure on W . Let π_ν be the representation of V on $\mathcal{S}(W \times X')$ given by*

$$\pi_\nu(v)(f)(z, w) := \theta(\langle \nu(z), v \rangle) f(z, w + p(v)).$$

Then

- (i) π_ν is acyclic

(ii) Note that $\nu(X'_0) \subset W^* \subset V^*$. Let $\nu_0 : X'_0 \rightarrow W^*$ be the map given by restriction of ν . Let $F \in \mathcal{T}(X'_0 \times W)$ be given by $F(z, w) := \langle \nu_0(z), w \rangle$. Consider the map $I_\nu : \mathcal{S}(W \times X') \rightarrow \mathcal{S}(X'_0)$ given by $I_\nu(f) := (pr_{X'})_*(f|_{X'_0 \times W} \cdot F)$, where $pr_{X'} : W \times X' \rightarrow W$ denotes the projection. Then I_ν defines an isomorphism $H_0(\mathfrak{v}, \pi_\nu) \xrightarrow{\sim} \mathcal{S}(X'_0)$.

Proof. (i) By the relative Shapiro Lemma (Lemma 10.2.1) it is enough to show that $\tau_{X', \nu}$ (see Notation 10.2.3) is acyclic. This follows from Lemma 10.2.2.

(ii) Fix a Haar measure on L . Since we fixed a Haar measure on W , this defines a Haar measure on V as well. Define $\bar{F} \in \mathcal{T}(X' \times V)$ by $\bar{F}(z, v) := \langle \nu(z), v \rangle$. Define $\bar{I} : \mathcal{S}(X' \times V) \rightarrow \mathcal{S}(X')$ by $\bar{I}(f) = (pr_{X'})_*(f) \cdot \bar{F}$. Let $r : \mathcal{S}(X') \rightarrow \mathcal{S}(X'_0)$ denote the restriction. Let $\mu : X' \rightarrow L^*$ be the composition of ν with $V^* \rightarrow L^*$. Define an action $\tilde{\pi}_\nu$ of V on $\mathcal{S}(X' \times V)$ given by

$$\pi_\nu(v)(f)(z, v') := \theta(\langle \nu(z), v \rangle) f(z, v' + v).$$

Define an action σ of L on $\mathcal{S}(X' \times V)$ by $\sigma(l)(f)(z, v') := f(z, v' + l)$.

Note that the following diagram is commutative

$$\begin{array}{ccccc}
 (1) & \mathfrak{v} \otimes \mathcal{S}(X' \times V) & \xrightarrow{Id_{\mathfrak{v}} \otimes (Id_{X'} \times p)_*} & \mathfrak{v} \otimes \mathcal{S}(W \times X') & \\
 & \downarrow d\tilde{\pi}_\nu & & \downarrow d\pi_\nu & \\
 (2) & \mathfrak{l} \otimes \mathcal{S}(X' \times V) & \xrightarrow{d\sigma} & \mathcal{S}(X' \times V) & \xrightarrow{(Id_{X'} \times p)_*} \mathcal{S}(W \times X') \longrightarrow 0 \\
 & \downarrow Id_{\mathfrak{l}} \otimes \bar{I} & & \downarrow \bar{I} & \downarrow I \\
 (3) & \mathfrak{l} \otimes \mathcal{S}(X') & \xrightarrow{\tau_\mu} & \mathcal{S}(X') & \xrightarrow{r} \mathcal{S}(X'_0) \longrightarrow 0 \\
 & \downarrow & & \downarrow 0 & \downarrow 0
 \end{array}$$

Recall that τ_μ , used in the diagram, is defined in Notation 10.2.3. It is enough to show that column (3) is exact. First, let us show that column (2) is exact. The map \bar{I} is onto by Lemma 9.6.14 and the exactness in the place of $\mathcal{S}(X' \times V)$ follows from 9.6.15. The exactness of row (2) is proven in the same way. The exactness of row (3) follows from Lemma 10.2.2. The exactness of column (2) implies the exactness of column (1). Let us prove that column (3) is exact. First, note that I is onto by Lemma 9.6.14 and Theorem 9.6.10(iii).

Now, let $f \in \text{Ker}(I) \subset \mathcal{S}(W \times X')$. Let $\tilde{f} \in \mathcal{S}(X' \times V)$ be its preimage under $(Id \times p)_*$. Then $\bar{I}(\tilde{f}) \in \text{Ker}(r) = \text{Im}(\tau_\mu)$. Let $h \in \mathfrak{l} \otimes \mathcal{S}(X')$ be a preimage of $\bar{I}(\tilde{f})$ and let \tilde{h} be a preimage of h in $\mathfrak{l} \otimes \mathcal{S}(X' \times V)$. Then $d\sigma(\tilde{h}) - \tilde{f} \in \text{Ker}(\bar{I}) = \text{Im } d\tilde{\pi}_\nu$. Now,

$$f = (Id \times p)_*(\tilde{f}) = (Id \times p)_*(\tilde{f} - d\sigma(\tilde{h}) + d\sigma(\tilde{h})) = (Id \times p)_*(\tilde{f} - d\sigma(\tilde{h})) \in (Id \times p)_*(\text{Im } d\tilde{\pi}_\nu) \subset \text{Im}(d\pi_\nu)$$

□

We will prove the following Lemma which clearly implies the analytic key lemma.

Lemma 10.3.2. *Let T be a Nash linear group and let $R := P_n \times T$ and $R' := G_{n-1} \times T$. Let $Q < R$ be a Nash subgroup and let $X \subset R/Q$ be a V_n -invariant open Nash subset and $X' := V_n \setminus X$. Note that X' is an open Nash subset of R'/Q' , where $Q' = Q/(Q \cap V_n)$. Let $X_0 = \{x \in X : \psi|_{(V_n)_x} = 1\}$. Let X'_0 be the image of X_0 in X' . Let \mathcal{E}' be a R' -equivariant tempered bundle on R'/Q' and $\mathcal{E} := p_{X'}^*(\mathcal{E}'|_{X'})$. Then*

- (i) $H_i(\mathfrak{v}_n, \mathcal{S}(X, \mathcal{E}) \otimes (-\psi)) = 0$ for any $i > 0$.
- (ii) X'_0 is smooth
- (iii) Since $X' \subset R'/Q'$ and $X \subset R/Q$ we have a canonical section to p_Z and will consider X' as a closed submanifold of X . Consider the action map $a : X'_0 \times V_n \rightarrow X_0$. Let $\Xi = 1 \boxtimes \psi \in \mathcal{T}(X'_0 \times V_n)$. Note that Ξ_0 is constant along the fibers of a . Define $\Xi \in \mathcal{T}(X_0)$ such that $a^*(\Xi) = \Xi_0$. Consider the map

$I : \mathcal{S}(X, p^? \mathcal{E}) \rightarrow \mathcal{S}(X'_0, \mathcal{E}|_{X'_0})$ given by $I(f) := (p_{X'}|_{X'_0})_*(f|_{X'_0} \cdot \Xi)$. Then I gives an isomorphism $H_0(\mathfrak{v}_n, S(X, p^? \mathcal{E})) \xrightarrow{\sim} \mathcal{S}(X'_0, \mathcal{E}|_{X'_0})$, as representations of \mathfrak{p}_{n-1} .

Remark 10.3.3. The open subset $X \subset R/Q$ is not necessary P_{n-1} -invariant, but \mathfrak{p}_{n-1} will still act on the homology.

First let us prove the following special case.

Lemma 10.3.4. Lemma 10.3.2 holds under the assumption that there exists a Nash section $a : X' \rightarrow R'$ of the quotient map $R' \rightarrow R'/Q'$.

Proof. Consider R'/Q' as a subset of R/Q . Without loss of generality we assume that the class of the unity element lies in X' . We will denote it by z_0 . Without loss of generality we assume $a(z_0) = 1$. Let $W := p_{X'}^{-1}(z_0)$. Let $L := (V_n)_{i(z_0)} = Q \cap V_n$. It is a Nash subgroup of V_n and hence is a real vector space. Note that $W \cong V_n/L$. Define $\phi : X' \times V_n \rightarrow X$ by $\phi(z, v) := (a(z)v)z$, where $a(z)v$ denotes the action of $a(z) \in G_{n-1}$ on $v \in V_n$. Note that ϕ factors through $X' \times W$ and let $\bar{\phi}$ be the corresponding map $X' \times W \rightarrow X$. It is easy to see that $\bar{\phi}$ is a Nash diffeomorphism and intertwines $\rho_{a, X', W}$ with the action of V_n on X (see Notation 10.2.6).

Note that the section a gives a tempered isomorphism between $\mathcal{E}'|_{X'}$ and the trivial bundle with fiber \mathcal{E}'_{z_0} . Thus $\bar{\phi}$ gives an V_n -isomorphism $\bar{\phi}^* : \mathcal{S}(X, \mathcal{E}) \otimes (-\psi) \xrightarrow{\sim} \pi_{a, X', W} \otimes \mathcal{E}'_{z_0}$. Define $\nu : X' \rightarrow V_n^*$ by $\nu(z)(v) := \bar{\psi}(a(z)v)$. Note that X'_0 coincides with X'_0 from Lemma 10.3.1. Note that the underlying vector space of $\pi_{a, X', W}$ coincides with the underlying vector space of π_ν from Lemma 10.3.1. Thus we can consider $I_\nu \otimes Id$ as a map $\pi_{a, X', W} \otimes \mathcal{E}'_{z_0} \rightarrow \mathcal{S}(X'_0) \otimes \mathcal{E}'_{z_0} \cong \mathcal{S}(X'_0, \mathcal{E}'|_{X'_0})$. Note that $\bar{\phi}^*$ intertwines the map I to $I_\nu \otimes Id$.

By Proposition 10.2.5, $H_*(\mathfrak{v}_n, \pi_{a, X', W}) \cong H_*(\mathfrak{v}_n, \pi_\nu)$ and on H_0 the isomorphism is just equality of quotient spaces. Thus it is enough to show that π_ν is acyclic and I_ν defines an isomorphism $H_0(\mathfrak{v}_n, \pi_\nu) \xrightarrow{\sim} \mathcal{S}(X'_0)$. To show this, by Lemma 10.3.1 it is enough to show that 0 is a regular value of the composition $\nu_L : X' \xrightarrow{\nu} V_n^* \rightarrow L^*$. This follows from Lemma 10.2.4. \square

Proof of Lemma 10.3.2. Let $q : R' \rightarrow R'/Q'$ be the quotient map. By Theorem 9.6.3, there exists a finite Nash cover $\{U_i\}$ of $X' \subset R'/Q'$ and sections $s_i : U_i \rightarrow R'$ of q . By Lemma 10.3.4, $\mathcal{S}(p_{X'}^{-1}(U_i), \mathcal{E}|_{p_{X'}^{-1}(U_i)})$ is acyclic and $I|_{\mathcal{S}(p_{X'}^{-1}(U_i), \mathcal{E}|_{p_{X'}^{-1}(U_i)})}$ gives an isomorphism

$$H_0(\mathfrak{v}_n, \mathcal{S}(p_{X'}^{-1}(U_i), \mathcal{E}|_{p_{X'}^{-1}(U_i)})) \xrightarrow{\sim} \mathcal{S}(U_i \cap X'_0, \mathcal{E}'|_{U_i \cap X'_0}).$$

Consider the extended Kozsul complex of $\mathcal{S}(X, \mathcal{E}) \otimes (-\psi)$:

$$(20) \quad 0 \rightarrow \Lambda^n(\mathfrak{v}_n) \otimes \mathcal{S}(X) \otimes (-\psi) \xrightarrow{d_n} \dots \xrightarrow{d_2} \mathfrak{v}_n \otimes \mathcal{S}(X) \otimes (-\psi) \xrightarrow{d_1} \mathcal{S}(X) \otimes (-\psi) \xrightarrow{d_0 := I} \mathcal{S}(X'_0, \mathcal{E}'|_{X'_0}) \rightarrow 0$$

We have to show that it is exact. The fact that I is onto follows from Theorem 9.6.10(iii) and Lemma 9.6.14. Let us show that it is exact in place $l \geq 0$, i.e. at the object $\Lambda^l(\mathfrak{v}_n) \otimes \mathcal{S}(X)$. Choose a partition of unity e_i corresponding to the cover $X' = \bigcup U_i$.

Let $\alpha \in \text{Ker } d_l \subset \Lambda^l(\mathfrak{v}_n) \otimes \mathcal{S}(X) \otimes (-\psi)$. Consider

$$p^*(e_i)\alpha \in \text{Ker } d_l|_{\Lambda^l(\mathfrak{v}_n) \otimes \mathcal{S}(p^{-1}(U_i) \otimes (-\psi))}.$$

By Lemma 10.3.4, we have

$$p^*(e_i)\alpha \in \text{Im}(d_{l+1}|_{\Lambda^l(\mathfrak{v}_n) \otimes \mathcal{S}(p^{-1}(U_i) \otimes (-\psi))}) \text{ and thus } \alpha = \sum p^*(e_i)\alpha \in \text{Im}(d_{l+1}).$$

\square

10.4. Untwisting a product (Proof of Proposition 10.2.5).

Lemma 10.4.1. *Let \mathfrak{g} be a Lie algebra and A be a commutative algebra with 1. Let M be a $(U(\mathfrak{g}) \otimes A)$ -module. Then $H_*(\mathfrak{g}, M) = H_*(\mathfrak{g} \otimes A, M)$.*

Proof. Let \mathcal{A} be the category of $(U(\mathfrak{g}) \otimes A)$ -modules and \mathcal{B} be the category of \mathfrak{g} -modules. Let $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ be the forgetful functor and $\mathcal{G} : \mathcal{B} \rightarrow Vect$ be the functor of \mathfrak{g} co-invariants. Note that $\mathcal{G} \circ \mathcal{F}$ is the functor of coinvariants with respect to the Lie algebra $\mathfrak{g} \otimes A$. By Lemma 9.2.5 we have

$$L^i(\mathcal{G} \circ \mathcal{F}) = L^i(\mathcal{G}) \circ \mathcal{F}.$$

This proves the assertion. \square

Proof of Proposition 10.2.5. Extend the representations π_i to representations Π_i of $\mathcal{T}(X) \otimes \mathfrak{g} \cong \mathcal{T}(X, \mathfrak{g})$.

Let $b : \mathcal{T}(X, \mathfrak{g}) \rightarrow \mathcal{T}(X, \mathfrak{g})$ be given by $b(f)(x) = a(x)(f(x))$. Note that b is invertible. Let us show that $\Pi_1 \cong \Pi_2 \circ b$.

First, note that as linear spaces both can be identified with $\mathcal{S}(X \times Y)$. Now, denote by $d\rho_i$ the corresponding maps from \mathfrak{g} to the space of vector fields on $X \times Y$ and by $d\mathfrak{X} : \mathfrak{g} \times X \times Y$ the differential of \mathfrak{X} in the first variable. Let $f \in \mathcal{T}(X, \mathfrak{g})$, $h \in \mathcal{S}(X \times Y)$, and $(x, y) \in X \times Y$. Then

$$\begin{aligned} (\Pi_1(f)h)(x, y) &= (d\rho_1(f(x))h)(x, y) + \chi(f(x))h(x, y) = (d\rho_2(a(x)(f(x)))h)(x, y) + \chi(f(x))h(x, y) = \\ &= (d\rho_2(a(x)(f(x)))h)(x, y) + \chi(a(x)^{-1}(a(x)(f(x)))) \cdot h(x, y) = \\ &= (d\rho_2(a(x)(f(x)))h)(x, y) + d\mathfrak{X}(a(x)(f(x)), x, y) \cdot h(x, y) = \\ &= (d\rho_2((b(f))(x))h)(x, y) + d\mathfrak{X}(b(f)(x), x, y) \cdot h(x, y) = (\Pi_2(b(f))(h))(x, y) \end{aligned}$$

Now

$$H_*(\mathfrak{g}, \pi_2) = H_*(\mathcal{T}(X) \otimes \mathfrak{g}, \Pi_2) \cong H_*(\mathcal{T}(X) \otimes \mathfrak{g}, \Pi_1) = H_*(\mathfrak{g}, \pi_1).$$

\square

10.5. Homology of families of characters (Proof of Lemma 10.2.2). We prove the lemma by induction on $\dim V$.

Base: $\dim V = 1$. In this case we have to show that the following extended Koszul complex is exact:

$$0 \leftarrow \mathcal{S}(X_0) \xleftarrow{r} \mathcal{S}(X) \leftarrow V \otimes \mathcal{S}(X) \leftarrow 0.$$

Let $v \in V$ be a generator. Then the complex is isomorphic to

$$0 \leftarrow \mathcal{S}(X_0) \xleftarrow{r} \mathcal{S}(X) \xleftarrow{m_h} \mathcal{S}(X) \leftarrow 0,$$

where $m_h(f) = hf$, and $h(x) = d\theta(\langle v, \phi(x) \rangle)$. Clearly, m_h is injective. The map r is onto by Theorem 9.6.10(iii). Note that ih is a real-valued Nash function and 0 is its regular value. The exactness in the middle follows now from Lemma 9.6.16.

Induction step. Let us first prove (i). Let $L < V$ be a one-dimensional subspace. Let ϕ_L denote the composition $X \rightarrow V^* \rightarrow L^*$. Note that 0 is a regular value of ϕ_L and let $X_{L,0} := \phi_L^{-1}(0)$. By the induction base, $\mathcal{S}(X)$ is \mathfrak{l} -acyclic and $r_L : H_0(\mathfrak{l}, \mathcal{S}(X)) \xrightarrow{\sim} \mathcal{S}(X_{L,0})$. Note that this is an isomorphism of representations of V/L . Note also that $\phi(X_{L,0}) \subset (V/L)^* = L^\perp \subset V^*$ and let ϕ' denote $\phi|_{X_{L,0}} : X_{L,0} \rightarrow (V/L)^*$. Note that 0 is a regular value of ϕ' . Finally, note that the action of V/L on $\mathcal{S}(X_{L,0})$ is $\tau_{X_{L,0}, \phi'}$. Thus, by the induction hypothesis, $H_0(\mathfrak{l}, \mathcal{S}(X))$ is V/L -acyclic and Lemma 9.2.7 implies that $\mathcal{S}(X)$ is V -acyclic.

To prove (ii) note first that r is onto by Theorem 9.6.10(iii). Now we have to show that $\mathfrak{v}\mathcal{S}(X) = \text{Ker } r$. The inclusion \subset is obvious. Let $f \in \text{Ker } r$. Consider $f|_{X_{L,0}}$. As before, the induction hypothesis implies $f|_{X_{L,0}} \in (\mathfrak{v}/\mathfrak{l})\mathcal{S}(X_{L,0})$. Let $h \in (\mathfrak{v}/\mathfrak{l}) \otimes \mathcal{S}(X_{L,0}) = \tau_{X_{L,0}, \phi'}((\mathfrak{v}/\mathfrak{l}) \otimes \mathcal{S}(X_{L,0}))$ such that $f|_{X_{L,0}} = \tau_{X_{L,0}, \phi'}(h)$. By Theorem 9.6.10(iii), we may extend h to $\tilde{h} \in \mathfrak{v} \otimes \mathcal{S}(X)$. Now, $(\tau_{X, \phi}(\tilde{h}) - f)|_{X_L} = 0$ and hence by the induction base $\tau_{X, \phi}(\tilde{h}) - f \in \mathfrak{l}\mathcal{S}(X)$. This implies that $f \in \mathfrak{v}\mathcal{S}(X)$. \square

10.6. Proof of Lemma 10.2.4. For the proof we will need the following straightforward lemma from linear algebra.

Lemma 10.6.1. *Let W_1, W_2 and W be linear spaces. Let $T : W_1 \rightarrow W_2$ and $A : W_1 \rightarrow W$ be epimorphisms. Suppose that $\text{Ker } T \supset \text{Ker } A$. Then $A \circ S$ is onto, for any section S of T .*

Proof of Lemma 10.2.4. Let $x \in U$ such that $\mu(x) = 0$. Without loss of generality, using the left action of G we can suppose $s(x) = 1$. Then $\phi|_L = 0$ and we have to show $d_{1a} \circ d_{xs}$ is onto L^* . The map d_{1a} is onto by transitivity of the action of G on $V^* \setminus 0$, and $\text{Ker } d_{1p} \subset \text{Ker } d_{1a}$ since Q stabilizes L . Thus, by the previous lemma, $d_{1a} \circ d_{xs}$ is onto. \square

APPENDIX A. PROOF OF THE RELATIVE SHAPIRO LEMMA

We prove here Theorem 10.2.1 which is a version of the Shapiro lemma. A similar version was proven in [AG10] and our proof follows the lines of the proof there.

Proposition A.0.1 ([AG10], Proposition 4.0.6). *Let G be a Nash group and X be a Nash G -manifold. Suppose that the action is strictly simple (see Definition 9.6.7). Then the projection $\pi : X \rightarrow G \setminus X$ is a Nash locally trivial fibration.*

Corollary A.0.2. *Let G be a Nash group and X be a Nash G -manifold with strictly simple action. Let $F \rightarrow X$ be a Nash G -equivariant locally-trivial fibration. Then the action of G on the total space F is strictly simple.*

Proof. By the previous proposition we may assume without loss of generality that the map $X \rightarrow G \setminus X$ is a trivial fibration. Hence we can identify $X \cong (G \setminus X) \times G$. Now it is easy to see that $G \setminus F \cong F|_{(G \setminus X) \times \{1\}}$. \square

Corollary A.0.3. *Let G be a Nash group and X be a Nash G -manifold. Suppose that the action is strictly simple. Let \mathcal{E} be a Nash G -equivariant bundle on X . Then there exists a Nash bundle \mathcal{E}' on $G \setminus X$ such that $\mathcal{E} = \pi^* \mathcal{E}'$, where $\pi : X \rightarrow G \setminus X$ is the standard projection.*

Proof. Consider the action of G on the total space of \mathcal{E} . Denote $\mathcal{E}' := G \setminus \mathcal{E}$ and consider it as a Nash bundle over $G \setminus X$. It is easy to see that $\mathcal{E} = \pi^* \mathcal{E}'$. \square

Definition A.0.4. *Let X be a Nash manifold and F be a Nash locally trivial fibration over X . Denote by $H_c^i(F \rightarrow X)$ the natural Nash bundle on X such that for any $x \in X$ we have $H_c^i(F \rightarrow X)_x = H_c^i(F_x)$. For its precise definition see [AG10, Notation 2.4.11]. Note that if F is a trivial fibration then $H_c^i(F \rightarrow X)$ are trivial bundles.*

Definition A.0.5. *Let $F \xrightarrow{\pi} X$ be a locally trivial fibration. Let $\mathcal{E} \rightarrow X$ be a Nash bundle. We define $T_{F \rightarrow X} \subset T_F$ by $T_{F \rightarrow X} = \text{Ker}(d\pi)$. We denote*

$$\Omega_{F \rightarrow X}^{i, \mathcal{E}} := ((T_{F \rightarrow X})^*)^{\wedge i} \otimes \pi^* \mathcal{E}$$

Now we define the relative de-Rham complexes $DR_{\mathcal{S}}^{\mathcal{E}}(F \rightarrow X)$ and $DR_{C^\infty}^{\mathcal{E}}(F \rightarrow X)$ by

$$DR_{\mathcal{S}}^{\mathcal{E}}(F \rightarrow X)^i := \mathcal{S}(F, \Omega_{F \rightarrow X}^{i, \mathcal{E}}) \text{ and } DR_{C^\infty}^{\mathcal{E}}(F \rightarrow X)^i := C^\infty(F, \Omega_{F \rightarrow X}^{i, \mathcal{E}}).$$

The differentials in those complexes are defined as the differential in the classical de-Rham complex. If \mathcal{E} is trivial we will omit it.

Theorem A.0.6 (see [AG10], Theorem 3.2.3). *Let $\pi : F \rightarrow X$ be an affine Nash locally trivial fibration. Then*

$$H^k(DR_{\mathcal{S}}^{\mathcal{E}}(F \rightarrow X)) \cong \mathcal{S}(X, H_c^k(F \rightarrow X) \otimes \mathcal{E}).$$

This theorem gives us the following recipe for computing Lie algebra homology.

Theorem A.0.7. *Let G be an affine Nash group. Let K be a Nash G -manifold and L be a Nash manifold. Let $X := K \times L$. Let $\mathcal{E} \rightarrow X$ be a Nash G -equivariant bundle. Let $\mathfrak{X} : G \times L \rightarrow \mathbb{R}$ be a Nash map such that for any $l \in L$, the map $\mathfrak{X}|_{G \times \{l\}}$ is a group homomorphism. Let \mathfrak{X}' be the 1-dimensional G -equivariant*

bundle on L , with action of G given by $g(l, v) = (l, \psi(\mathfrak{X}(g))v)$. Let $\mathcal{E}' := \mathcal{E} \otimes (\mathbb{C} \boxtimes \mathfrak{X}')$. Note that \mathcal{E}' is isomorphic to \mathcal{E} as a Nash bundle, though the G -equivariant structure on \mathcal{E}' is not necessarily Nash.

Let N be a strictly simple Nash G -manifold. Suppose that N and G are homologically trivial and affine. Denote $F = X \times N$. Note that the bundle $\mathcal{E}' \boxtimes \Omega_N^i$ has G -equivariant structure given by diagonal action. Hence the relative de-Rham complex $DR_{C^\infty}^{\mathcal{E}'}(F \rightarrow X)$ is a complex of representations of G . Consider the relative de-Rham complex $DR_{\mathcal{S}}^{\mathcal{E}'}(F \rightarrow X)$ without the action of G as a subcomplex of $DR_{C^\infty}^{\mathcal{E}'}(F \rightarrow X)$. Note that it is G -invariant. We denote by $DR_{\mathcal{S}}^{\mathcal{E}'}(F \rightarrow X)$ the complex $DR_{\mathcal{S}}^{\mathcal{E}'}(F \rightarrow X)$ with the action of G induced from $DR_{C^\infty}^{\mathcal{E}'}(F \rightarrow X)$.

Then

$$H_i(\mathfrak{g}, \mathcal{S}(X, \mathcal{E}')) = H^{n-i}((DR_{\mathcal{S}}^{\mathcal{E}'}(F \rightarrow X))_{\mathfrak{g}}),$$

where n is the dimension of X .

To prove this theorem we will need the following statements.

Lemma A.0.8. *Let $G, L, N, \mathfrak{X}, \mathfrak{X}'$ be as in Theorem A.0.7. Let X be a Nash manifold. Let \mathcal{E} be a Nash bundle over $X \times L \times N$. Let $\mathcal{E}' = \mathcal{E} \otimes (\mathbb{C} \boxtimes \mathfrak{X}' \boxtimes \mathbb{C})$. Then $\mathcal{S}(X \times L \times N, \mathcal{E}) \simeq \mathcal{S}(X \times L \times N, \mathcal{E}')$ as representations of G .*

Proof. By Theorem 9.6.10(iv) and Proposition A.0.1 we can assume that $N = G \times N'$, for some Nash manifold N' . Hence we can assume $N = G$. Note that $\mathcal{S}(X \times L \times G, \mathcal{E})$ and $\mathcal{S}(X \times L \times G, \mathcal{E}')$ are identical as linear spaces. Now, the required isomorphism between them is given by $\alpha \mapsto (1 \boxtimes (\psi \circ \mathfrak{X}))\alpha$. \square

Lemma A.0.9. *Let G be an affine Nash group. Let F be a strictly simple Nash G -manifold. Denote $X := G \setminus F$. Let $\mathcal{E} \rightarrow X$ be a Nash bundle. Then the relative de-Rham complex $DR_{\mathcal{S}}^{\mathcal{E}'}(F \rightarrow X)^i$ is isomorphic to the complex $\tilde{C}(\mathfrak{g}, \mathcal{S}(F, \pi^* \mathcal{E}))^{\dim \mathfrak{g} - i}$, where $\pi : F \rightarrow X$ is the standard projection, and $\tilde{C}(\mathfrak{g}, W)$ denotes the Koszul complex of a representation W .*

The proof of this lemma is the same as the proof of [AG10, Lemma 4.0.10].

Corollary A.0.10. *Let an affine Nash group G act strictly simply on a Nash manifold X . Let \mathcal{E} be a Nash G -equivariant bundle on X . Suppose that G is homologically trivial. Then $\mathcal{S}(X, \mathcal{E})$ is an acyclic representation of \mathfrak{g} .*

Proof. Follows from Lemma A.0.9, Theorem A.0.6 and Corollary A.0.3. \square

Proof of Theorem A.0.7. From Theorem A.0.6 we know that the complex $DR_{\mathcal{S}}^{\mathcal{E}'}(F \rightarrow X)$, as a complex of vector spaces, is a resolution of the vector space $\mathcal{S}(X, \mathcal{E})$. Hence the complex $DR_{\mathcal{S}}^{\mathcal{E}'}(F \rightarrow X)$ is a resolution of $\mathcal{S}(X, \mathcal{E}')$ as a representation of \mathfrak{g} .

So it is enough to prove that the representations $\mathcal{S}(F, \mathcal{E}' \boxtimes \Omega_N^i)$ are \mathfrak{g} -acyclic. By Lemma A.0.8,

$$\mathcal{S}(F, \mathcal{E} \boxtimes \Omega_N^i) \cong \mathcal{S}(F, \mathcal{E}' \boxtimes \Omega_N^i)$$

as \mathfrak{g} -representations (though those isomorphisms do not commute with differentials). Hence it is enough to show that $\mathcal{S}(F, \mathcal{E} \boxtimes \Omega_N^i)$ is acyclic. This follows from Corollary A.0.10. \square

Lemma A.0.11 ([AG], Corollary B.1.8). *Let G be a connected affine Nash group, X be a Nash manifold and \mathcal{E} be a Nash bundle over X . Let \mathfrak{g} be the Lie algebra of G . Let $p : G \times X \rightarrow X$ be the projection. Let G act on $\mathcal{S}(G \times X, p^*(\mathcal{E}))$ by acting on the G coordinate. Then the map $p_* : \mathcal{S}(G \times X, p^*(\mathcal{E})) \rightarrow \mathcal{S}(X, \mathcal{E})$ induces an isomorphism $\mathcal{S}(G \times X, p^*(\mathcal{E}))_{\mathfrak{g}} \cong \mathcal{S}(X, \mathcal{E})$.*

Corollary A.0.12. *Let G be a connected affine Nash group. Let F be a strictly simple Nash G -manifold. Denote $X := G \setminus F$. Let $\mathcal{E} \rightarrow X$ be a Nash bundle. Fix a Haar measure on G and a Nash measure on X . These give rise to a Nash measure on F . Let π denote the projection map $\pi : F \rightarrow X$. Let B be a bundle on X such that $\mathcal{E} = \pi^* B$ (B exists by Lemma A.0.3).*

Then the map $\pi_ : \mathcal{S}(F, \mathcal{E} \otimes D_X^F) \rightarrow \mathcal{S}(X, B)$ induces an isomorphism $\mathcal{S}(F, \mathcal{E})_{\mathfrak{g}} \cong \mathcal{S}(X, B)$.*

Corollary A.0.13. *Let X be a Nash manifold and Y be a closed Nash submanifold. Let an affine Nash group G act on X strictly simply and H be a subgroup of G that acts on Y strictly simply. Suppose that the natural map $H \setminus Y \rightarrow G \setminus X$ is a Nash diffeomorphism. Suppose that G and H are homologically trivial. Let \mathcal{E} be a Nash G -equivariant bundle on X . Then*

$$\mathcal{S}(X, \mathcal{E} \otimes D_X)_{\mathfrak{g}} \cong \mathcal{S}(Y, \mathcal{E}|_Y \otimes D_Y)_{\mathfrak{h}}.$$

Moreover, let $a : G \times Y \rightarrow X$ denote the action map and $p_2 : G \times Y \rightarrow Y$ denote the projection on the second coordinate. Then

(i) The map a_* gives an isomorphism

$$a_* : \mathcal{S}(G \times Y, (\mathbb{C} \boxtimes \mathcal{E}|_Y) \otimes D_{G \times Y} \otimes a^*(D_X^{-1}))_{\mathfrak{g} \times \mathfrak{h}} \xrightarrow{\sim} \mathcal{S}(X, \mathcal{E})_{\mathfrak{g}}$$

(ii) The map $(p_2)_*$ gives an isomorphism

$$(p_2)_* : \mathcal{S}(G \times Y, (\mathbb{C} \boxtimes \mathcal{E}|_Y) \otimes D_{G \times Y} \otimes a^*(D_X^{-1}))_{\mathfrak{g} \times \mathfrak{h}} \xrightarrow{\sim} \mathcal{S}(Y, \mathcal{E}|_Y \otimes D_Y \otimes ((D_X)|_Y)^{-1})_{\mathfrak{h}}$$

Corollary A.0.14. *Let G be an affine Nash group and X be a transitive Nash G -manifold. Let N be a Nash manifold. Let $x \in X$ and denote $H := G_x$. Suppose that G and H are homologically trivial. Let $X = X \times N \times G$ and $Y = \{x\} \times N \times G$. Let \mathcal{E} be a bundle over X . Let $\mathfrak{X} : G \times N \rightarrow \mathbb{R}$ be a Nash map such that for any $n \in N$, the map $\mathfrak{X}|_{G \times \{n\}}$ is a group homomorphism. Let \mathfrak{X}' be the 1-dimensional G -equivariant bundle on N , with the action of G given by $g(n, v) = (n, \psi(\mathfrak{X}(g))v)$. Let $\mathcal{E}' := \mathcal{E} \otimes (\mathbb{C} \boxtimes \mathfrak{X}' \boxtimes \mathbb{C})$. Then*

(i) The map a_* gives an isomorphism

$$a_* : \mathcal{S}(G \times Y, (\mathbb{C} \boxtimes \mathcal{E}'|_Y) \otimes D_{G \times Y} \otimes a^*(D_X^{-1}))_{\mathfrak{g} \times \mathfrak{h}} \cong \mathcal{S}(X, \mathcal{E}')_{\mathfrak{g}}$$

(ii) The map $(p_2)_*$ gives an isomorphism

$$(p_2)_* : \mathcal{S}(G \times Y, (\mathbb{C} \boxtimes \mathcal{E}'|_Y) \otimes D_{G \times Y} \otimes a^*(D_X^{-1}))_{\mathfrak{g} \times \mathfrak{h}} \cong \mathcal{S}(Y, \mathcal{E}'|_Y \otimes D_Y \otimes ((D_X)|_Y)^{-1})_{\mathfrak{h}}$$

Now we are ready to prove the relative Shapiro Lemma.

Proof of Theorem 10.2.1. From Theorem A.0.7 we see that

$$H_i(\mathfrak{g}, \mathcal{S}(X \times N, \mathcal{E}')) \cong H^{\dim \mathfrak{g} - i}((DR_{\mathcal{S}}^{\mathcal{E}'}(X \times N \times G \rightarrow X \times N))_{\mathfrak{g}}).$$

Now, by Corollary A.0.14,

$$\mathcal{S}(X \times N \times G, \Omega_{X \times N \times G \rightarrow X \times N}^{i, \mathcal{E}'}|_{\mathfrak{g}}) \cong \mathcal{S}(\{x\} \times N \times G, \Omega_{\{x\} \times N \times G \rightarrow \{x\} \times N}^{i, \mathcal{E}'|_{\{x\} \times N} \otimes D_{\{x\} \times N} \otimes D_{X \times N}^{-1}|_{\{x\} \times N}})_{\mathfrak{h}}.$$

and this isomorphism commutes with de-Rham differential. Therefore

$$(DR_{\mathcal{S}}^{\mathcal{E}'}(X \times N \times G \rightarrow X \times N))_{\mathfrak{g}} \cong (DR_{\mathcal{S}}^{\mathcal{E}'|_{\{x\} \times N} \otimes D_{\{x\} \times N} \otimes D_{X \times N}^{-1}|_{\{x\} \times N}}(\{x\} \times N \times G \rightarrow \{x\} \times N))_{\mathfrak{h}}$$

and hence

$$H^{\dim \mathfrak{g} - i}((DR_{\mathcal{S}}^{\mathcal{E}'}(X \times N \times G \rightarrow X \times N))_{\mathfrak{g}}) \cong H^{\dim \mathfrak{g} - i}((DR_{\mathcal{S}}^{\mathcal{E}'|_{\{x\} \times N} \otimes (D_X^{-1}|_{\{x\}} \boxtimes \mathbb{C})}(\{x\} \times N \times G \rightarrow \{x\} \times N))_{\mathfrak{h}})$$

and again by Theorem A.0.7,

$$\begin{aligned} H^{\dim \mathfrak{g} - i}((DR_{\mathcal{S}}^{\mathcal{E}'|_{\{x\} \times N} \otimes (D_X^{-1}|_{\{x\}} \boxtimes \mathbb{C})}(\{x\} \times N \times G \rightarrow \{x\} \times N))_{\mathfrak{h}}) &\cong \\ &\cong H_i(\mathfrak{h}, \mathcal{S}(\{x\} \times N, \mathcal{E}'|_{\{x\} \times N} \otimes (D_X^{-1}|_{\{x\}} \boxtimes \mathbb{C}))) \cong \\ &\cong H_i(\mathfrak{h}, \mathcal{S}(\{x\} \times N, \mathcal{E}'|_{\{x\} \times N} \otimes \Delta_H \cdot (\Delta_G^{-1})|_H)). \end{aligned}$$

□

APPENDIX B. PROOF OF PROPOSITION 5.2.7

The current formulation of the proposition and the proof written down in this Appendix are results of a conversation of one of the authors with J. Bernstein. After writing it down we learnt that Proposition 5.2.7 was proven by O. Gabber and written down by A. Joseph in his lecture notes [Jos80]. We decided to keep this Appendix for the convenience of the reader.

We will need some lemmas on filtrations.

Definition B.0.1. *Two filtrations F and Φ are called comparable if there exists $k > 0$ such that for all $i \geq 0$ we have $\Phi^i M \subset \mathbb{F}^{i+k} M \subset \Phi^{i+2k} M$.*

Lemma B.0.2. *Suppose F and Φ are comparable. Then there exist filtrations Ψ_j , $0 \leq j \leq 2k$ such that $\Psi_0^i M = F^{i+k} M$, $\Psi_{2k}^i M = \Phi^{i+2k} M$, and $\Psi_j^i M \subset \Psi_{j+1}^i M \subset \Psi_j^{i+1} M \subset \Psi_{j+1}^{i+1} M$.*

Proof. Define $\Psi_j^i M := F^{i+k} M + \Phi^{i+j} M$. □

Lemma B.0.3. *Any two good filtrations are comparable.*

Corollary B.0.4. *Let Φ and Ψ be 2 good filtrations on M . Then $\text{Gr}_\Phi(M)$ and $\text{Gr}_\Psi(M)$ are Jordan-Hölder equivalent. In particular, $\text{AV}_\Phi(M)$ does not depend on the choice of good filtration Φ .*

Lemma B.0.5. *Let F and Φ be two filtrations on M such that $F^i M \subset \Phi^i M \subset \mathbb{F}^{i+1} M \subset \Phi^{i+1} M$ for any $i \geq 0$. Consider the natural morphism $\phi : \text{Gr}_F(M) \rightarrow \text{Gr}_\Phi(M)$ and let $K := \text{Ker } \phi$, and $C := \text{Coker } \Phi$. Then C is isomorphic to the graded $\text{Gr}(A)$ -module $K[1]$ obtained from K by shifting the gradation by one.*

Proof. The i -th grade of ϕ is the natural morphism $F^i M / F^{i-1} M \rightarrow \Phi^i M / \Phi^{i-1} M$. Therefore $K_i = \Phi^{i-1} M / F^{i-1} M$ and $C_i = \Phi^i M / F^i M$. □

Corollary B.0.6. *Suppose $\text{Gr}(A)$ is Noetherian. Let F and Φ be two comparable filtrations on M . Then F is a good filtration if and only if Φ is a good filtration.*

Proof. By Lemma B.0.2, we may suppose that $F^i M \subset \Phi^i M \subset \mathbb{F}^{i+1} M \subset \Phi^{i+1} M$ for any $i \geq 0$. Recall that a filtration is good if and only if the associated graded module is finitely generated. Consider the exact sequence of $\text{Gr}(A)$ -modules $0 \rightarrow K \rightarrow \text{Gr}_F(M) \rightarrow \text{Gr}_\Phi(M) \rightarrow C$ from the previous lemma. Suppose that F is a good filtration. Then $\text{Gr}_F(M)$ is finitely generated and hence Noetherian. Thus so is K , and by the previous lemma so is C . Hence $\text{Gr}_\Phi(M)$ is also finitely generated and hence Φ is good. The other implication is proven in a similar way. □

The proof of Proposition 5.2.7 will be based on the following proposition from [Wall88, §§3.7].

Proposition B.0.7 (Casselman-Osborne). *Let \mathfrak{n} denote the nilradical of the complexified Lie algebra of the minimal parabolic subgroup of G and \mathfrak{k} denote the complexified Lie algebra of the maximal compact subgroup of G . Let $Z_G(\mathfrak{g}) := U(\mathfrak{g})^G$. Then*

There exists a finite-dimensional subspace $E \subset U(\mathfrak{g})$ such that

$$U(\mathfrak{g})^i \subset U(\mathfrak{n})^i EZ_G(\mathfrak{g})U(\mathfrak{k}) \text{ and therefore } U(\mathfrak{g}) = U(\mathfrak{n})EZ_G(\mathfrak{g})U(\mathfrak{k})$$

Proof of Proposition 5.2.7. Let $\pi \in \mathcal{M}_{HC}(G)$. Let us first construct one \mathfrak{n} -good \mathfrak{g} -filtration.

Let $V \subset \pi$ be a finite dimensional K -invariant and Z_G -invariant generating subspace. Let $E \subset U(\mathfrak{g})$ be such that $U(\mathfrak{g})^i \subset U(\mathfrak{n})^i EZ_G(\mathfrak{g})U(\mathfrak{k})$, as in Proposition B.0.7. Define $F^i := U(\mathfrak{g})^i V$ and $\Phi^i := U(\mathfrak{n})^i EV = U(\mathfrak{n})^i EZ_G(\mathfrak{g})U(\mathfrak{k})V$. Note that Φ^i is a good \mathfrak{n} -filtration and that $F^i \subset \Phi^i \subset F^{i+k}$, where k is the maximal degree of an element in E . Thus, by Corollary B.0.6, F^i is a good \mathfrak{n} -filtration.

Now, let f^i be any good \mathfrak{g} -filtration on π . By Lemma B.0.3 f^i is comparable to F^i and thus, by Corollary B.0.6, f^i is \mathfrak{n} -good. □

APPENDIX C. EXAMPLES FOR THE NOTATION OF §§8.3

In order to help the reader to read Notations 8.3.1 and 8.4.2, Lemma 8.3.2 and Corollary 8.3.4, we describe explicitly the objects discussed in them on one example. Let $n = 6$ and $\lambda = (1, 3, 2)$. Then $m_\lambda^1 = 1$, $m_\lambda^2 = 4$, $m_\lambda^3 = 6$. Also,

$$P_\lambda = \left\{ \begin{pmatrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \end{pmatrix} \in G_6 \right\}, \quad c = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$w_1 = Id_6, \quad w_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad w_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\overline{P_n w_1 P_\lambda} = P_n w_1 P_\lambda = \left\{ \begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * \end{pmatrix} \in G_6 \right\}$$

$$\overline{P_n w_2 P_\lambda} = P_n w_2 P_\lambda \cup P_n w_1 P_\lambda = \left\{ \begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & * & * & * & * & * \end{pmatrix} \in G_6 \right\}$$

$$\overline{P_n w_2 P_\lambda} = P_n w_1 P_\lambda \cup P_n w_2 P_\lambda \cup P_n w_3 P_\lambda = G_6$$

$$P_\lambda^1 = P_\lambda \quad P_\lambda^2 = \left\{ \begin{pmatrix} * & * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * \\ * & * & 0 & * & * & * \\ * & * & 0 & * & * & * \\ * & * & 0 & * & * & * \end{pmatrix} \in G_6 \right\}, \quad P_\lambda^3 = \left\{ \begin{pmatrix} * & * & * & * & * & 0 \\ * & * & * & * & * & 0 \\ * & * & * & * & * & 0 \\ 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & * & * & 0 \\ * & * & * & * & * & * \end{pmatrix} \in G_6 \right\}$$

$$Q_\lambda^1 = \left\{ \begin{pmatrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in G_6 \right\} \quad Q_\lambda^2 = \left\{ \begin{pmatrix} * & * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * \\ * & * & 0 & * & * & * \\ * & * & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in G_6 \right\}$$

$$Q_\lambda^3 = \left\{ \begin{pmatrix} * & * & * & * & 0 \\ * & * & * & * & 0 \\ * & * & * & * & 0 \\ 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in G_6 \right\}$$

In order to help the reader to read Lemma 8.3.6, we describe explicitly the objects discussed there for $i = 2$. Let $\lambda' = \lambda_{3-2+1}^- = \lambda_2^- = (1, 2, 2)$. Then

$$w_{\lambda'}^{21} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad w_{\lambda'}^{22} = Id, \quad w_{\lambda'}^{23} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

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